The Rothe Method for Variational-Hemivariational Inequalities with Applications to Contact Mechanics

Krzysztof Bartosz, Mircea Sofonea

THE ROTHE METHOD FOR VARIATIONAL-HEMIVARIATIONAL INEQUALITIES WITH APPLICATIONS TO CONTACT MECHANICS*

KRZYSZTOF BARTOSZ† AND MIRCEA SOFONEA‡

Abstract. We consider a new class of first order evolutio nary variational-hemivariational in-equalities for which we prove an existence and uniqueness res ult. The proof is based on a time-discretization method, also known as the Rothe method. It cons ists of considering a discrete version of each inequality in the class, proving its unique solvability, an d recovering the solution of the con-tinuous problem as the time step converges to zero. Then we introduce a quasi-static frictionless problem for Kelvin–Voigt viscoelastic materials in which the con tact is modeled with a nonmonotone normal compliance condition and a unilateral constraint. We prove the variational formulation of the problem cast in the abstract setting of variational-hemivariati onal inequalities, with a convenient choice of spaces and operators. Further, we apply our abstract result in order to prove the unique weak solvability of the problem.

Key words. variational-hemivariational inequality, Clarke subdifferential, Rothe method, existence and uniqueness, viscoelastic material, frictionless contact, normal compliance, unilateral constraint, weak solution

1. Introduction. Variational and hemivariational inequalities play an important role in the study of both the qualitative and numerical analysis of various problems arising in mechanics, physics, and the engineering sciences. Basic references in the field include [2, 7, 8, 11, 12, 23, 25, 28, 30, 31]. Started in the early 1960s, the theory of variational inequalities uses as its main ingredients the arguments of monotonicity and convexity, including properties of the subdifferential of a convex function. In contrast, the theory of hemivariational inequalities is based on properties of the subdifferential in the sense of Clarke, defined for locally Lipschitz functions which may be nonconvex. The numerical analysis of variational inequalities is a broadly developed field, as illustrated in [14, 15, 20] and the references therein. In contrast, there are still very few publications devoted to numerical methods for hemivariational inequalities and, in particular, to evolutionary hemivariational inequalities. The basic reference in the field is the book [19]. The Rothe method represents one of the few discretization methods used in the analysis of variational and hemivariational inequalities. It was considered in [21], for instance, in the study of parabolic hemivariational inequalities. The results obtained there were generalized in [22], where

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†Faculty of Mathematics and Computer Science, Jagiellonian University, Institute of Computer Science, 30348 Krakow, Poland (krzysztof.jakub.bartosz@gmail.com). This author's research was partially supported by the International Project co-financed by the Ministry of Science and Higher Education of Republic of Poland under grant W111/7.PR/2012.

‡Corresponding author. Laboratoire de Mathématiques et Physique, Université de Perpignan Via Domitia, 66860 Perpignan, France (sofonea@univ-perp.fr).

a θ-discretization scheme was considered. The Rothe method was also considered in [6], in the study of parabolic variational-hemivariational inequality. References concerning the numerical treatment of hemivariational inequalities arising in contact mechanics include [3, 4, 5].

Variational-hemivariational inequalities represent a special class of inequalities, in which both convex and nonconvex functions are involved. Interest in their study is motivated by various problems in mechanics, as shown in [28, 29]. Recent results in their study have been obtained in [16, 26, 33]. The inequalities studied in [16] are elliptic. There, an existence and uniqueness result was proved through arguments of surjectivity for pseudomonotone operators and the Banach fixed point. The analysis of a class of history-dependent variational-hemivariational inequalities, including their unique solvability, was provided in [26]. There, the proof was based on arguments on pseudomonotone operators, again, combined with a fixed point result for nonlinear operators defined on the space of continuous functions. The analysis of the quasi-variational inequalities introduced in [26] was continued in [33], where a continuous dependence result was proved, numerical schemes to solve the inequalities were obtained, and error estimates were derived. The history-dependent variational-hemivariational inequalities considered in [16, 26, 33] were formulated in the particular case of Sobolev spaces associated to a bounded domain $\Omega \subset \mathbb{R}^d$ and to specific operators like the trace operator, for instance.

The aim of this paper is to study a new class of evolutionary variational-hemivariational inequalities with applications to contact mechanics. With respect to our previous works [26, 33] this paper has some traits of novelty that we describe in what follows. First, in contrast with the inequalities considered in [26, 33], the inequalities we study in this current paper are introduced in the context of abstract reflexive Banach spaces. Second, they are evolutionary; i.e., they involve the derivative of the unknown function. In addition, their unique weak solvability is obtained by using arguments different from those in [26], based on the Rothe method. And, finally, we apply these results to a new model of contact for Kelvin–Voigt viscoelastic materials.

The rest of the paper is organized as follows. In section 2 we review some preliminary material on nonlinear analysis. In section 3 we introduce the class of evolutionary variational-hemivariational inequalities to be studied, list the assumptions on the data, and state our main existence and uniqueness result, Theorem 3.1. The proof of the theorem is presented in section 5. It is based on the Rothe method that we describe in section 4. Next, in section 6, we introduce a frictionless contact problem in which the material's behavior is modeled with the Kelvin–Voigt viscoelastic constitutive law and the contact conditions are with normal compliance and unilateral constraints. We list the assumptions on the data and derive the weak formulation of the problem, which is in the form of a variational-hemivariational inequality for the displacement field. Finally, we apply our abstract result to prove the unique weak solvability of the viscoelastic contact problem.

2. Notation and preliminaries. In this section we briefly present the notation and some preliminary material to be used later in this paper. More details on the material presented below can be found in the books [10, 11, 25, 28, 35].

First, we make precise that all linear spaces used in this paper are assumed to be real. Unless it is stated otherwise, in this section we denote by X a normed space with the norm $\|\cdot\|_X$, we denote by X^* its topological dual, and $\langle \cdot, \cdot \rangle_{X^* \times X}$ will represent the duality pairing of X and X^*. The symbol 2^{X^*} is used to represent the set of all subsets of X^*. We start with the definition of the subdifferential in the sense of

Clarke.

DEFINITION 2.1. *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function. The Clarke generalized directional derivative of φ at the point $x \in X$ in the direction $v \in X$ is defined by*

$$\varphi^0(x;v) = \limsup_{y\to x,\lambda\downarrow 0} \frac{\varphi(y+\lambda v)-\varphi(y)}{\lambda}.$$

The Clarke subdifferential of φ at x is a subset of X^* given by

$$\partial_{Cl}\varphi(x) = \{\, \zeta \in X^* : \varphi^0(x;v) \geq \langle \zeta, v\rangle_{X^*\times X} \quad \text{for all} \quad v \in X \,\}.$$

For a convex function we also recall the definition of its subdifferential in the sense of convex analysis.

DEFINITION 2.2. *Let $\Phi : X \to \mathbb{R}\cup\{+\infty\}$ be a convex functional. Then the convex subdifferential of Φ at $x \in X$ is a subset of X^* given by*

$$\partial_{Conv}\Phi(x) = \{\, \xi \in X^* : \Phi(x+v) - \Phi(x) \geq \langle \xi, v\rangle_{X^*\times X} \quad \text{for all } v \in X \,\}.$$

Now we pass to the definition of pseudomonotonicity, for both single valued and multivalued operators.

DEFINITION 2.3. *A single valued operator $A : X \to X^*$ is called pseudomonotone if for any sequence $\{v_n\}_{n=1}^\infty \subset X$, $v_n \to v$ weakly in X and*

$$\limsup_{n\to\infty} \langle Av_n, v_n - v\rangle_{X^*\times X} \leq 0$$

imply that $\langle Av, v-y\rangle_{X^*\times X} \leq \liminf_{n\to\infty} \langle Av_n, v_n - y\rangle_{X^*\times X}$ for every $y \in X$.

DEFINITION 2.4. *A multivalued operator $A : X \to 2^{X^*}$ is called pseudomonotone if the following conditions hold:*

(1) *A has values which are nonempty, weakly compact, and convex.*
(2) *A is upper semicontinuous (usc, for short) from every finite dimensional subspace of X into X^* endowed with the weak topology.*
(3) *For any sequence $\{v_n\}_{n=1}^\infty \subset X$ and any $v_n^* \in A(v_n)$, $v_n \to v$ weakly in X and $\limsup_{n\to\infty} \langle v_n^*, v_n - v\rangle_{X^*\times X} \leq 0$ imply that for any $y \in X$ there exists $u(y) \in A(v)$ such that $\langle u(y), v-y\rangle_{X^*\times X} \leq \liminf_{n\to\infty} \langle v_n^*, v_n - y\rangle_{X^*\times X}$.*

The next proposition provides the pseudomonotonicity of a multivalued operator corresponding to a superposition of the Clarke subdifferential with a compact operator. For its proof we refer the reader to the proof of Proposition 5.6 in [17].

PROPOSITION 2.5. *Let X and U be two reflexive Banach spaces, let $\iota\colon X \to U$ be a linear, continuous, and compact operator, and denote by $\iota^* : U^* \to X^*$ the adjoint operator of ι. Let $J\colon U \to \mathbb{R}$ be a locally Lipschitz functional, and assume that its Clarke subdifferential satisfies*

$$\|\xi\|_{U^*} \leq c\,(1 + \|v\|_U) \quad \text{for all } \xi \in \partial J(v)$$

with $c > 0$. Then the multivalued operator $M : X \to 2^{X^*}$ defined by

$$M(v) = \iota^*\partial J(\iota v) \quad \text{for all } v \in X$$

is pseudomonotone.

Note that, in the statement of Proposition 2.5, U^* represents the dual of U and $\|\cdot\|_U$ and $\|\cdot\|_{U^*}$ denote the norms on the spaces U and U^*, respectively.

We now recall a result providing pseudomonotonicity of the sum of two pseudomonotone operators, which corresponds to Proposition 1.3.68 in [11].

PROPOSITION 2.6. *Assume that X is a reflexive Banach space and $A_1, A_2 : X \to 2^{X^*}$ are pseudomonotone operators. Then $A_1 + A_2 : X \to 2^{X^*}$ is a pseudomonotone operator.*

The next proposition deals with an existence result for an abstract elliptic inclusion and corresponds to Theorem 2.2 in [24].

PROPOSITION 2.7. *Let X be a real reflexive Banach space, let $F\colon D(F) \subset X \to 2^{X^*}$ be a maximal monotone operator, let $G : D(G) = X \to 2^{X^*}$ be a multivalued pseudomonotone operator, and let $L \in X^*$. Assume that there exist $u_0 \in X$ and $R \geq \|u_0\|_X$ such that $D(F) \cap B_R(0_X) \neq \emptyset$ and*

$$\langle \xi + \eta - L, u - u_0 \rangle_{X^* \times X} > 0 \tag{2.1}$$

for all $u \in D(F)$ with $\|u\|_X = R$ and all $\xi \in F(u)$, $\eta \in G(u)$. Then there exists at least an element $u \in D(F)$ such that

$$F(u) + G(u) \ni L. \tag{2.2}$$

Note that in the statement of Proposition 2.7 we denote by $D(F)$ and $D(G)$ the effective domains of the operators F and G, respectively, 0_X represents the zero element of X, and $B_R(0_X)$ represents the ball of radius R and center 0_X.

We now introduce some spaces of vector functions defined on the interval $[0, T]$ where $T > 0$. Let π denote a finite partition of the interval $(0, T)$ by a family of disjoint subintervals $\sigma_i = (a_i, b_i)$ such that $[0, T] = \cup_{i=1}^n \bar{\sigma}_i$. Let $\mathcal{F}$ denote the family of all such partitions. Then for $1 \leq q < \infty$ we define the seminorm of a function $x : [0, T] \to X$ by equality

$$\|x\|^q_{BV^q(0,T;X)} = \sup_{\pi \in \mathcal{F}} \Big\{ \sum_{\sigma_i \in \pi} \|x(b_i) - x(a_i)\|^q_X \Big\}$$

and the space

$$BV^q(0, T; X) = \{\, x : [0, T] \to X;\ \|x\|_{BV^q(0;T;X)} < \infty \,\}.$$

Assume now that $1 \leq p \leq \infty$, $1 \leq q < \infty$, and X, Z are Banach spaces such that $X \subset Z$ with continuous embedding. Denote

$$M^{p,q}(0, T; X, Z) = L^p(0, T; X) \cap BV^q(0, T; Z).$$

Then it is well known that $M^{p,q}(0, T; X, Z)$ is also a Banach space with the norm $\|\cdot\|_{L^p(0,T;X)} + \|\cdot\|_{BV^q(0,T;Z)}$.

We end this section with the following compactness result obtained in [21].

PROPOSITION 2.8. *Let $1 \leq p, q < \infty$. Let $X_1 \subset X_2 \subset X_3$ be real Banach spaces such that X_1 is reflexive, the embedding $X_1 \subset X_2$ is compact, and the embedding $X_2 \subset X_3$ is continuous. Then the embedding $M^{p,q}(0, T; X_1; X_3) \subset L^p(0, T; X_2)$ is compact.*

We shall use Proposition 2.8 in section 6 of this paper.

3. An existence and uniqueness result. In this section, we present the variational-hemivariational inequality in which we are interested, list the assumptions on the data, and state our main existence and uniqueness result, Theorem 3.1.

Let V be a strictly convex, reflexive separable Banach space, and let V^* denote its dual. We denote by $\langle \cdot, \cdot \rangle_{V^* \times V}$ and $\| \cdot \|_V$ the duality pairing between V and V^* and the norm on V, respectively. Let $\iota : V \to U$, where U is a reflexive Banach space, and let $\langle \cdot, \cdot \rangle_{U^* \times U}$ and $\| \cdot \|_U$ denote the duality pairing between U and U^* and the norm on U, respectively. We also use the notation $\mathcal{L}(V, V^*)$ for the space of linear continuous operators from V to V^*, and we denote by $\| \cdot \|_{\mathcal{L}(V,V^*)}$ the norm in space $\mathcal{L}(V, V^*)$. Analogously, we introduce the space $\mathcal{L}(V, U)$ and the corresponding norm $\| \cdot \|_{\mathcal{L}(V,U)}$. For $T > 0$ we use the classical notation for the Lebesgue and Sobolev spaces of function defined on the interval $[0, T]$ with values in a normed space. In addition, we introduce the notation

$$
\begin{aligned}
&\mathcal{V} = L^2(0, T; V), \quad \mathcal{V}^* = L^2(0, T; V^*),\\
&\mathcal{U} = L^2(0, T; U), \quad \mathcal{U} = L^2(0, T; U^*),\\
&\mathcal{W} = \{\, u \in \mathcal{V} \mid u' \in \mathcal{V}^* \,\},
\end{aligned}
$$

where u' represents the time derivative of u understood in the sense of distributions. Moreover, we use the symbols $\langle \cdot, \cdot \rangle_{\mathcal{V}^* \times \mathcal{V}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{U}^* \times \mathcal{U}}$ to denote the duality pairing between $\mathcal{V}$ and $\mathcal{V}^*$, $\mathcal{U}$ and $\mathcal{U}^*$, respectively. In addition, $C(0; T; V)$ will represent the space of continuous functions defined on $[0, T]$ with values in V, endowed with its usual norm.

Let $A, B : V \to V^*$ be given operators, let $J : U \to \mathbb{R}$, $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ be given functionals, and let $f : [0, T] \to V^*$ be a given function. With these data we consider the following problem.

Problem $\mathcal{P}$. *Find $u \in \mathcal{W}$ such that $u(0) = u_0$ and*

$$
\begin{aligned}
\text{(3.1)} \qquad &\int_0^T \langle Au'(t) + Bu(t) + \iota^* \xi(t) - f(t), v(t) - u(t) \rangle_{V^* \times V}\, dt \\
&\qquad + \int_0^T \big(\Phi(v(t)) - \Phi(u(t)) \big)\, dt \geq 0 \quad \text{for all } v \in \mathcal{V}
\end{aligned}
$$

with

$$
\text{(3.2)} \qquad \xi(t) \in \partial_{Cl} J(\iota u(t)) \qquad \text{for a.e. } t \in (0, T).
$$

In the study of Problem $\mathcal{P}$ we consider the following assumptions on the data:

$H(A)$. The operator $A : V \to V^*$ is linear, bounded, coercive, and symmetric, i.e., the following hold:

(i) $A \in \mathcal{L}(V, V^*)$.
(ii) $\langle Av, v \rangle_{V^* \times V} \geq \alpha \|v\|_V^2$ for all $v \in V$ with $\alpha > 0$.
(iii) $\langle Av, w \rangle_{V^* \times V} = \langle Aw, v \rangle_{V^* \times V}$ for all $v, w \in V$.

$H(B)$. The operator $B : V \to V^*$ is linear, bounded, and coercive, i.e., the following hold:

(i) $B \in \mathcal{L}(V, V^*)$.
(ii) $\langle Bv, v \rangle_{V^* \times V} \geq \beta \|v\|_V^2$ for all $v \in V$ with $\beta > 0$.

$H(J)$. The functional $J : U \to \mathbb{R}$ is such that the following hold:

(i) J is locally Lipschitz.
(ii) $\partial_{Cl} J$ satisfies the growth condition $\|\xi\|_{U^*} \leq c\,(1 + \|u\|_U)$ for all $u \in U$ and all $\xi \in \partial_{Cl} J(u)$ with $c > 0$.

(iii) There exists $m > 0$ such that

$$\langle \xi - \eta, u - v\rangle_{U^*\times U} \geq -m\|u - v\|_U^2$$

for all $u,\, v \in U,\ \xi \in \partial_{Cl} J(u),\ \eta \in \partial_{Cl} J(v)$.

$H(\Phi)$. The functional $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ is convex, proper, and lower semicontinuous.

$H(\iota)$. The operator $\iota : V \to U$ is linear, continuous, and compact. Moreover, the associated Nemytskii operator $\bar{\iota} : M^{2,2}(0,T;V,V^*) \to \mathcal{U}$ defined by $(\bar{\iota}v)(t) = \iota(v(t))$ for all $t \in [0,T]$ is also compact.

$H(0)$. $f \in H^1(0,T;V^*)$, $u_0 \in dom(\Phi)$, and the following compatibility condition holds: there exist $\xi_0 \in \partial_{Cl} J(\iota u_0)$ and $\eta_0 \in \partial_{Conv}\Phi(u_0)$ such that

$$Bu_0 + \iota^*\xi_0 + \eta_0 - f(0) \in V.$$

$H(s)$. Inequality $\beta > m$ holds, where β and m represent the constants introduced in assumptions $H(B)$ and $H(J)$(iii), respectively.

Note that here and below we use the notation $dom(\Phi)$ for the effective domain of the function Φ. Also, we recall that condition $H(j)$(iii) represents the so-called relaxed monotonicity condition, intensively used in the study of various classes of hemivariational inequalities, as shown in [25] and the references therein.

The main result of this paper that we state here and prove in section 5 is the following.

THEOREM 3.1. *Assume that $H(A)$, $H(B)$, $H(J)$, $H(\Phi)$, $H(\iota)$, $H(0)$, and $H(s)$ hold. Then Problem $\mathcal{P}$ has a unique solution $u \in H^1(0,T;V)$.*

We end this section with the remark that Problem $\mathcal{P}$ is constructed by using the convex function Φ and the nonconvex function J. In addition, it involves the derivative of the solution. For this reason, we refer to Problem $\mathcal{P}$ as an evolutionary variational-hemivariational inequality.

4. The Rothe problem. In this section we consider an approximate problem based on time discretization of Problem $\mathcal{P}$. For this discrete problem, also known as the Rothe problem, we prove a result of solvability and we derive error estimates for the solution. To this end, let $N \in \mathbb{N}$ be fixed and let $\tau = \frac{T}{N}$ represent the time step. We consider a piecewise constant approximation of f given by

$$f_\tau(t) = f_\tau^k := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} f(s)\,ds \ \text{ for } \ t \in ((k-1)\tau, k\tau]\,, \quad \text{for } \ k = 1, \ldots, N. \tag{4.1}$$

Then, using Lemma 3.3 from [9], we know that

$$f_\tau \to f \ \text{ strongly in } \ \mathcal{V}^* \ \text{ as } \ \tau \to 0. \tag{4.2}$$

We now formulate the following Rothe problem.

PROBLEM $\mathcal{P}_\tau$. *Find a sequence $\{u_\tau^k\}_{k=0}^N \subset V$ such that $u_\tau^0 = u_0$ and*

$$\begin{aligned} &\left\langle \frac{1}{\tau} A(u_\tau^k - u_\tau^{k-1}), v - u_\tau^k \right\rangle_{V^*\times V} + \langle Bu_\tau^k, v - u_\tau^k\rangle_{V^*\times V} + \langle \xi_\tau^k, \iota(v - u_\tau^k)\rangle_{U^*\times U} \\ &\qquad + \Phi(v) - \Phi(u_\tau^k) \geq \langle f_\tau^k, v - u_\tau^k\rangle_{V^*\times V} \quad \text{for all } v \in V \end{aligned} \tag{4.3}$$

with

$$\xi_\tau^k \in \partial_{Cl} J(\iota u_\tau^k) \tag{4.4}$$

for $k = 1, \ldots, N$.

We have the following existence result.

LEMMA 4.1. *Assume that* $H(A), H(B), H(J), H(\Phi)$, *and* $H(0)$ *hold. Then there exists* $\tau_1 > 0$ *such that Problem* $\mathcal{P}_\tau$ *has at least one solution for all* $\tau \in (0, \tau_1)$.

Proof. First we observe that Problem $\mathcal{P}_\tau$ can be formulated in the following equivalent way: given $u_\tau^{k-1} \in V$ with $k = 1, \ldots, N-1$, find $u_\tau^k \in V$ such that

$$\frac{1}{\tau} A u_\tau^{k-1} + f_\tau^k \in \partial_{Conv} \Phi(u_\tau^k) + \frac{1}{\tau} A u_\tau^k + B u_\tau^k + \iota^* \partial_{Cl} J(\iota u_\tau^k). \tag{4.5}$$

In order to establish the solvability of (4.5) we apply Proposition 2.7 with $F(u) = \partial_{Conv} \Phi(u)$ and $G(u) = \frac{1}{\tau} Au + Bu + \iota^* \partial_{Cl} J(\iota u)$. To this end we observe that F is a maximal monotone operator, since it represents the subdifferential of the function Φ which is proper, convex, and lower semicontinuous. Moreover, since the operators $\frac{1}{\tau} A$ and B are linear and monotone, they are pseudomonotone. Then, from Propositions 2.5 and 2.6, it follows that G is a pseudomonotone operator.

Let u_0 be the element used as the initial condition in Problem $\mathcal{P}_\tau$. Let u, ξ, η be such that $u \in D(F)$, $\xi \in F(u)$, $\eta \in G(u)$. We will show that, taking $L = \frac{1}{\tau} A u_\tau^{k-1} + f_\tau^k$, the inequality (2.1) holds for all u such that $\|u\|_V = R$, where $R \geq \|u_0\|_V$. We have

$$\eta = \frac{1}{\tau} Au + Bu + \iota^* w \quad \text{with } w \in \partial_{Cl} J(\iota u). \tag{4.6}$$

Our aim in what follows is to show that if $\|u\|_V$ is large enough, then the following inequality holds:

$$\left\langle \frac{1}{\tau} Au + Bu + \iota^* w + \xi, u - u_0 \right\rangle_{V^* \times V} > \left\langle \frac{1}{\tau} A u_\tau^{k-1} + f_\tau^k, u - u_0 \right\rangle_{V^* \times V}. \tag{4.7}$$

To this end we note that, using our Definition 2.2 and Lemma 2.5 in [27], there exist $k_1, k_2 > 0$ such that

$$\langle \xi, u - u_0 \rangle_{V^* \times V} \geq \Phi(u) - \Phi(u_0) \geq -k_1 \|u\|_V - k_2 - \Phi(u_0) \quad \text{for all } u \in V. \tag{4.8}$$

Next, from $H(A)$, $H(B)$, $H(J)$, (4.8), and the inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, valid for $a, b, \varepsilon > 0$, we have

$$\begin{aligned} \left\langle \frac{1}{\tau} Au + Bu + \iota^* w + \xi, u - u_0 \right\rangle_{V^* \times V} &\geq \left(\frac{1}{\tau} \alpha + \beta - c \|\iota\|^2_{\mathcal{L}(V,U)} - 3\varepsilon \right) \|u\|_V^2 \\ &- \frac{1}{4\varepsilon} c^2 \|\iota\|^2_{\mathcal{L}(V,U)} - \frac{1}{4\varepsilon} \left(\frac{1}{\tau} \|A\|_{\mathcal{L}(V,V^*)} + \|B\|_{\mathcal{L}(V,V^*)} + c\, \|\iota\|_{\mathcal{L}(V,U)} \right)^2 \|u_0\|_V^2 \\ &- c\, \|\iota\|_{\mathcal{L}(V,U)} \|u_0\|_V - \frac{k_1^2}{4\varepsilon} - k_2 - \Phi(u_0). \end{aligned} \tag{4.9}$$

On the other hand, we have

$$\begin{aligned} &\left\langle \frac{1}{\tau} A u_\tau^{k-1} + f_\tau^k, u - u_0 \right\rangle_{V^* \times V} \\ &\leq \varepsilon \|u\|_V^2 + \left(\frac{1}{4\varepsilon} + \frac{1}{2} \right) \left\| \frac{1}{\tau} A u_\tau^{k-1} + f_\tau^k \right\|^2_{V^*} + \frac{1}{2} \|u_0\|_V^2. \end{aligned} \tag{4.10}$$

Let us denote $\delta(\tau) = \frac{1}{\tau}\alpha + \beta - c\|\iota\|^2_{\mathcal{L}(V,U)}$ and observe that if $\beta \geq c\|\iota\|^2_{\mathcal{L}(V,U)}$, then $\delta(\tau) > 0$ for all $\tau > 0$. Otherwise, if $\beta < c\|\iota\|^2_{\mathcal{L}(V,U)}$, we have $\delta(\tau) > 0$ for all $\tau < \tau_1 := \alpha(c\|\iota\|^2_{\mathcal{L}(V,U)} - \beta)^{-1}$. In both cases we see that for $\varepsilon = \frac{1}{8}\delta(\tau)$, the value $\delta(\tau) - 4\varepsilon$ is positive. Thus, there exists $R_1 > 0$ such that

$$
\begin{aligned}
(4.11)\qquad & \left(\frac{1}{\tau}\alpha + \beta - c\|\iota\|^2_{\mathcal{L}(V,U)} - 4\varepsilon\right)\|u\|^2_V \\
& > \left[\frac{1}{4\varepsilon}\left(\frac{1}{\tau}\|A\|_{\mathcal{L}(V,V^*)} + \|B\|_{\mathcal{L}(V,V^*)} + c\|\iota\|_{\mathcal{L}(V,U)}\right)^2 + \frac{1}{2}\right]\|u_0\|^2_V \\
& + c\,\|\iota\|_{\mathcal{L}(V,U)}\|u_0\|_V + \frac{k_1^2}{4\varepsilon} + k_2 + \Phi(u_0) + \frac{1}{4\varepsilon}c^2\|\iota\|^2_{\mathcal{L}(V,U)} \\
& + \left(\frac{1}{4\varepsilon} + \frac{1}{2}\right)\left\|\frac{1}{\tau}Au_\tau^{k-1} + f_\tau^k\right\|^2_{V^*} + \frac{1}{2}\|u_0\|^2_V
\end{aligned}
$$

for all $u \in V$ such that $\|u\|_V \geq R_1$. From (4.9)–(4.11) we conclude that (4.7) holds for all $u \in V$ satisfying $\|u\|_V \geq R_1$. Next, by $H(0)$, we have $\partial_{Conv}\Phi(u_0) \neq \emptyset$, so $u_0 \in D(F)$, and therefore $D(F) \cap B_{\|u_0\|_V}(0_X) \neq \emptyset$. We now remark that the constant $R = \max\{ R_1, \|u_0\|_V \}$ satisfies the assumptions of Proposition 2.7. We conclude from here that problem (4.5) has a solution, which completes the proof. □

We now derive an appropriate estimate for the solution of the Rothe problem.

Lemma 4.2. *Assume that $H(A), H(B), H(J), H(\Phi), H(0)$, and $H(s)$ hold. Let the sequence $\{u_\tau^k\}_{k=0}^N$ be a solution of Problem $\mathcal{P}_\tau$, obtained in Lemma 4.1. Then there exist $\tau_2 > 0$ and some positive constants c_1, c_2, c_3, c_4 independent on τ such that*

$$(4.12)\qquad \max_{k=1,\dots,N}\|u_\tau^k\|_V \leq c_1,$$

$$(4.13)\qquad \sum_{k=1}^N \|u_\tau^k - u_\tau^{k-1}\|^2_V \leq c_2,$$

$$(4.14)\qquad \max_{k=1\dots N}\left\|\frac{u_\tau^k - u_\tau^{k-1}}{\tau}\right\|_V \leq c_3,$$

$$(4.15)\qquad \tau\sum_{k=1}^N\left\|\frac{u_\tau^k - u_\tau^{k-1}}{\tau}\right\|^2_V \leq c_4$$

for all $\tau \in (0, \tau_2)$.

Proof. We take $v_0 \in dom(\Phi)$ as the test function in (4.3) and obtain

$$
\begin{aligned}
(4.16)\qquad & \left\langle \frac{1}{\tau}A\left(u_\tau^k - u_\tau^{k-1}\right), u_\tau^k\right\rangle_{V^*\times V} + \langle Bu_\tau^k, u_\tau^k\rangle_{V^*\times V} + \langle \xi_\tau^k, \iota u_\tau^k\rangle_{U^*\times U} + \Phi(u_\tau^k) \\
& \leq \left\langle \frac{1}{\tau}A\left(u_\tau^k - u_\tau^{k-1}\right), v_0\right\rangle_{V^*\times V} + \langle Bu_\tau^k, v_0\rangle_{V^*\times V} \\
& + \langle \xi_\tau^k, \iota v_0\rangle_{U^*\times U} + \Phi(v_0) + \langle f_\tau^k, u_\tau^k - v_0\rangle_{V^*\times V}
\end{aligned}
$$

with $\xi_\tau^k \in \partial_{Cl}J(\iota u_\tau^k)$ and $k = 1, \dots, N$.

From $H(A)$ we obtain

$$
\begin{aligned}
(4.17)\qquad & \langle A(u-v),u\rangle_{V^*\times V} = \frac{1}{2}\langle Au,u\rangle_{V^*\times V} - \frac{1}{2}\langle Av,v\rangle_{V^*\times V} \\
& + \frac{1}{2}\langle A(u-v),(u-v)\rangle_{V^*\times V} \quad \text{for all } u,v\in V.
\end{aligned}
$$

Then, using this identity, assumptions $H(B)$, $H(J)$, and Lemma 2.5 in [27], we obtain

$$
\begin{aligned}
(4.18)\qquad & \left\langle \frac{1}{\tau} A\left(u_\tau^k - u_\tau^{k-1}\right), u_\tau^k \right\rangle_{V^*\times V} + \langle Bu_\tau^k, u_\tau^k\rangle_{V^*\times V} + \langle \xi_\tau^k, \iota u_\tau^k\rangle_{U^*\times U} + \Phi(u_\tau^k) \\
& \geq \frac{1}{2\tau}\langle Au_\tau^k, u_\tau^k\rangle_{V^*\times V} - \frac{1}{2\tau}\langle Au_\tau^{k-1}, u_\tau^{k-1}\rangle_{V^*\times V} \\
& + \frac{1}{2\tau}\langle A\left(u_\tau^k - u_\tau^{k-1}\right), u_\tau^k - u_\tau^{k-1}\rangle_{V^*\times V} + \left(\beta - c\|\iota\|^2_{\mathcal{L}(V,U)} - 2\varepsilon\right)\|u_\tau^k\|_V^2 \\
& - \frac{1}{4\varepsilon}\left(c^2\|\iota\|_{\mathcal{L}(V,U)} + k_1\right) - k_2,
\end{aligned}
$$

where k_1, k_2 are positive constants which do not depend on τ. Moreover, we have the estimate

$$
\begin{aligned}
(4.19)\qquad & \left\langle \frac{1}{\tau} A\left(u_\tau^k - u_\tau^{k-1}\right), v_0 \right\rangle_{V^*\times V} + \langle Bu_\tau^k, v_0\rangle_{V^*\times V} + \langle \xi_\tau^k, \iota v_0\rangle_{U^*\times U} + \Phi(v_0) \\
& + \langle f_\tau^k, u_\tau^k - v_0\rangle_{V^*\times V} \leq \left\langle \frac{1}{\tau} A\left(u_\tau^k - u_\tau^{k-1}\right), v_0 \right\rangle_{V^*\times V} + 3\varepsilon\|u_\tau^k\|_V^2 \\
& + \frac{1}{4\varepsilon}\left(\|B\|^2_{\mathcal{L}(V,V^*)}\|v_0\|_V^2 + c^2\|\iota\|^4_{\mathcal{L}(V,U)}\|v_0\|_V^2\right) \\
& + \left(\frac{1}{4\varepsilon} + \frac{1}{2}\right)\|f_\tau^k\|^2_{V^*} + \Phi(v_0) + \frac{1}{2}\|v_0\|_V^2.
\end{aligned}
$$

Next, we combine (4.16)–(4.19) to find that

$$
\begin{aligned}
(4.20)\qquad & \frac{1}{2}\langle Au_\tau^k, u_\tau^k\rangle_{V^*\times V} + \frac{1}{2}\langle A\left(u_\tau^k - u_\tau^{k-1}\right), u_\tau^k - u_\tau^{k-1}\rangle_{V^*\times V} + \tau\beta\|u_\tau^k\|_V^2 \\
& \leq \langle A\left(u_\tau^k - u_\tau^{k-1}\right), v_0\rangle_{V^*\times V} + \frac{1}{2}\langle Au_\tau^{k-1}, u_\tau^{k-1}\rangle_{V^*\times V} \\
& + \tau\left(c\|\iota\|^2_{\mathcal{L}(V,U)} + 5\varepsilon\right)\|u_\tau^k\|_V^2 + \left(\frac{1}{4\varepsilon} + \frac{1}{2}\right)\tau\|f_\tau^k\|^2_{V^*} \\
& + \tau\frac{1}{4\varepsilon}\left(\|B\|^2_{\mathcal{L}(V,V^*)}\|v_0\|_V^2 + c^2\|\iota\|^4_{\mathcal{L}(V,U)}\|v_0\|_V^2 + c^2\|\iota\|_{\mathcal{L}(V,U)} + k_1\right) \\
& + \tau\left(\Phi(v_0) + \frac{1}{2}\|v_0\|_V^2 + k_2\right).
\end{aligned}
$$

We now introduce the notation

$$
\begin{aligned}
C_1(\varepsilon) &= \left(\frac{1}{4\varepsilon} + \frac{1}{2}\right), \\
C_2(\varepsilon) &= \frac{1}{4\varepsilon}\left(\|B\|^2_{\mathcal{L}(V,V^*)}\|v_0\|^2_V + c^2\|\iota\|^4_{\mathcal{L}(V,U)}\|v_0\|^2_V + c^2\|\iota\|_{\mathcal{L}(V,U)} + k_1\right) \\
&\quad + \Phi(v_0) + \frac{1}{2}\|v_0\|^2_V + k_2,
\end{aligned}
$$

write (4.20) for $k = 1, \dots, n \le N$, add the resulting inequalities, and use assumption $H(A)$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\alpha\|u^n_\tau\|^2_V + \frac{1}{2}\alpha\sum_{k=1}^{n}\|u^k_\tau - u^{k-1}_\tau\|^2_V + \tau\sum_{k=1}^{n}\beta\|u^k_\tau\|^2_V \le \epsilon\|u^n_\tau\|^2_V + \frac{1}{4\epsilon}\|A\|^2_{\mathcal{L}(V,V^*)}\|v_0\|^2_V \\
&+ \frac{1}{2}\|A\|_{\mathcal{L}(V,V^*)}\|u^0_\tau\|^2_V + \tau\sum_{k=1}^{n}\left(c\|\iota\|^2_{\mathcal{L}(V,U)} + 5\varepsilon\right)\|u^k_\tau\|^2_V + C_1(\varepsilon)\|f_\tau\|^2_{\mathcal{V}^*} + TC_2(\varepsilon).
\end{aligned}
$$

This inequality implies that

$$
\begin{aligned}
(4.21)\qquad &\left(\frac{1}{2}\alpha - \tau c\|\iota\|^2_{\mathcal{L}(V,U)} - \epsilon - 5\tau\varepsilon\right)\|u^n_\tau\|^2_V \le \tau\sum_{k=1}^{n-1}\left(c\|\iota\|^2_{\mathcal{L}(V,U)} + 5\varepsilon\right)\|u^k_\tau\|^2_V \\
&+ \frac{1}{4\epsilon}\|A\|^2_{\mathcal{L}(V,V^*)}\|v_0\|^2_V + \frac{1}{2}\|A\|_{\mathcal{L}(V,V^*)}\|u^0_\tau\|^2_V + C_1(\varepsilon)\|f_\tau\|^2_{\mathcal{V}^*} + TC_2(\varepsilon).
\end{aligned}
$$

Let $\tau_2 = \alpha(2c\|\iota\|^2_{\mathcal{L}(V,U)})^{-1}$, and assume that $\tau < \tau_2$. Then there exist $\epsilon, \varepsilon > 0$ such that $\frac{1}{2}\alpha - \tau c\|\iota\|^2_{\mathcal{L}(V,U)} - \epsilon - 5\tau\varepsilon > 0$. On the other hand, (4.2) implies that the sequence $\{f_\tau\}$ is bounded in $\mathcal{V}^*$ as $\tau \to 0$. Therefore, we are in a position to apply the discrete Gronwall lemma, i.e., Lemma 7.25 in [18]. As a result, from (4.21) we obtain (4.12). Moreover, (4.13) follows from (4.12) and (4.21).

We now proceed with the proof of inequalities (4.14) and (4.15). To this end we take $\xi_0 \in \partial_{Cl}J(u_0)$, $\eta_0 \in \partial_{Conv}\Phi(u_0)$ and define $u^{-1}_\tau = u_0 + \tau(Bu_0 + \iota^*\xi_0 + \eta_0 - f(0))$. Moreover, for $k = 0, \dots, N$, we denote $\delta u^k_\tau = \frac{1}{\tau}(u^k_\tau - u^{k-1}_\tau)$. Then it follows that

$$
(4.22)\qquad \delta u^0_\tau = f(0) - Bu_0 - \iota^*\xi_0 - \eta_0.
$$

Taking $v = u^k_\tau$ in (4.3) we obtain

$$
\begin{aligned}
&\tau\left\langle A\delta u^k_\tau, \delta u^k_\tau\right\rangle_{V^*\times V} + \tau\left\langle Bu^k_\tau, \delta u^k_\tau\right\rangle_{V^*\times V} + \tau\left\langle \xi^k_\tau, \iota\delta u^k_\tau\right\rangle_{U^*\times U} \\
&+ \Phi(u^k_\tau) - \Phi(u^{k-1}_\tau) \le \tau\left\langle f^k_\tau, \delta u^k_\tau\right\rangle_{V^*\times V}
\end{aligned}
$$

with

$$
\xi^k_\tau \in \partial_{Cl}J(u^k_\tau).
$$

Moreover, replacing k with $k-1$ in (4.3) and taking $v = u^k_\tau$ yields

$$
\begin{aligned}
&-\tau\left\langle A\delta u^{k-1}_\tau, \delta u^k_\tau\right\rangle_{V^*\times V} - \tau\left\langle Bu^{k-1}_\tau, \delta u^k_\tau\right\rangle_{V^*\times V} \\
&-\tau\left\langle \xi^{k-1}_\tau, \iota\delta u^k_\tau\right\rangle_{U^*\times U} + \Phi(u^{k-1}_\tau) - \Phi(u^k_\tau) \le \tau\left\langle f^{k-1}_\tau, \delta u^k_\tau\right\rangle_{V^*\times V}
\end{aligned}
$$

with

$$
\xi^{k-1}_\tau \in \partial_{Cl}J(u^{k-1}_\tau).
$$

We now add the last two inequalities and use (4.17), $H(B)$, $H(J)$(iii) to obtain

$$\text{(4.23)}\quad \frac{1}{2}\left\langle A\delta u_\tau^k, \delta u_\tau^k\right\rangle_{V^*\times V} + \frac{1}{2}\left\langle A\left(\delta u_\tau^k - \delta u_\tau^{k-1}\right), \delta u_\tau^k - \delta u_\tau^{k-1}\right\rangle_{V^*\times V}$$
$$+\tau(\beta - m - \varepsilon)\|\delta u_\tau^k\|_V^2 \le \frac{1}{2}\left\langle A\delta u_\tau^{k-1}, \delta u_\tau^{k-1}\right\rangle_{V^*\times V} + \frac{1}{4\varepsilon}\frac{1}{\tau}\|f_\tau^k - f_\tau^{k-1}\|_{V^*}^2.$$

We now write (4.23) for $k = 1, \ldots, n \le N$; then we add the resulting inequalities to obtain

$$\text{(4.24)}\quad \frac{1}{2}\alpha\,\|\delta u_\tau^n\|_V^2 + \sum_{k=1}^{n}\tau(\beta - m - \varepsilon)\|\delta u_\tau^k\|_V^2 \le \frac{1}{4\varepsilon}\,\|f'\|_{\mathcal{V}^*}^2$$
$$+ 2\,\|A\|_{\mathcal{L}(V,V^*)}\left(\|f(0)\|^2 + \|B\|_{\mathcal{L}(V,V^*)}^2\|u_0\|_V^2 + \|\iota\|_{\mathcal{L}(V,U)}^2\|\xi_0\|_V^2 + \|\eta_0\|_V^2\right).$$

Estimates (4.14) and (4.15) are now a direct consequence of (4.24) and assumption $H(s)$. ◻

5. Proof of Theorem 3.1. In this section we provide the proof of Theorem 3.1 and, to this end, we use the estimates obtained in section 4 in the study of the Rothe problem. Below we consider various sequences indexed by $\tau > 0$ for which we investigate their boundedness and convergence as $\tau \to 0$. We start by considering the piecewise linear and piecewise constant interpolant functions $u_\tau : [0,T] \to V$ and $\bar{u}_\tau : [0,T] \to V$, respectively, defined by

$$\text{(5.1)}\qquad \bar{u}_\tau(t) = \begin{cases} u_\tau^k, & t \in ((k-1)\tau, k\tau]\,,\quad k = 1, \ldots, N, \\ u_\tau^0, & t = 0, \end{cases}$$

and

$$\text{(5.2)}\qquad u_\tau(t) = u_\tau^k + \left(\frac{t}{\tau} - k\right)(u_\tau^k - u_\tau^{k-1}),\ \ t \in ((k-1)\tau, k\tau]\,,\quad k = 1, \ldots, N.$$

Here the sequence $\{u_\tau^k\}_{k=0}^N$ is a solution of Problem $\mathcal{P}_\tau$, obtained under the assumptions of Lemma 4.1. In addition, we consider the piecewise constant interpolant $\bar{\xi}_\tau : (0,T] \to U^*$ given by

$$\text{(5.3)}\qquad \bar{\xi}_\tau(t) = \xi_\tau^k,\ \ t \in ((k-1)\tau, k\tau]\,,\quad k = 1, \ldots, N,$$

where the sequence $\{\xi_\tau^k\}_{k=0}^N$ satisfies (4.4). Then we note that (4.3) can be written, equivalently, as

$$\text{(5.4)}\quad \langle Au_\tau'(t) + B\bar{u}_\tau(t) - f_\tau(t), v(t) - \bar{u}_\tau(t)\rangle_{V^*\times V} + \langle\bar{\xi}_\tau(t), \bar{\iota}(v(t) - \bar{u}_\tau(t))\rangle_{U^*\times U}$$
$$+\,\Phi(v(t)) - \Phi(\bar{u}_\tau(t)) \ge 0 \quad \text{for all } v \in \mathcal{V} \text{ a.e. } t \in (0,T).$$

We now define the Nemytskii operators $\mathcal{A}, \mathcal{B} : \mathcal{V} \to \mathcal{V}^*$ as $(\mathcal{A}w)(t) = A(w(t))$ and $(\mathcal{B}w)(t) = B(w(t))$ for all $w \in \mathcal{V}$ and all $t \in [0,T]$. Thus, from (5.4) we get

$$\text{(5.5)}\qquad \langle\mathcal{A}u_\tau' + \mathcal{B}\bar{u}_\tau - f_\tau, v - \bar{u}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} + \left\langle\bar{\xi}_\tau, \bar{\iota}(v - \bar{u}_\tau)\right\rangle_{\mathcal{U}^*\times\mathcal{U}}$$
$$+\int_0^T \Phi(v(t)) - \Phi(\bar{u}_\tau(t))\,dt \ge 0 \quad \text{for all } v \in \mathcal{V},$$

where, recall, $\bar{\iota}$ is the Nemytskii operator introduced in assumption $H(\iota)$. Let τ_2 be the constant obtained in the proof of Lemma 4.2. We have the following result.

Lemma 5.1. *Assume that $H(A), H(B), H(J), H(\Phi)$, $H(0)$, and $H(s)$ hold. Then, for all $\tau \in (0, \tau_2)$, the functions defined by (5.1)–(5.3) satisfy*

$$\|\bar{u}_\tau\|_{L^\infty(0,T;V)} \le d_1, \tag{5.6}$$

$$\|\bar{u}_\tau\|_{M^{2,2}(0,T;V,V^*)} \le d_2, \tag{5.7}$$

$$\|u_\tau\|_{C(0,T;V)} \le d_3, \tag{5.8}$$

$$\|u'_\tau\|_{\mathcal{V}} \le d_4, \tag{5.9}$$

$$\|\bar{\xi}_\tau\|_{\mathcal{U}^*} \le d_5 \tag{5.10}$$

with some positive constants d_i, $1 \le i \le 5$, not dependent on τ.

Proof. Since the assumptions of Lemma 4.2 are satisfied, the estimates (4.12)–(4.14) hold for all $\tau \in (0, \tau_2)$. The estimates (5.6) and (5.8) follow directly from (4.12). Next, we use (5.6) to see that the sequence $\{\bar{u}_\tau\}$ remains bounded in $\mathcal{V}$. Therefore, in order to establish (5.7), it is enough to estimate $\|\bar{u}_\tau\|_{BV^2(0,T;V^*)}$. To this end we consider a division $0 = a_0 < a_1 < \cdots < a_n = T$, where $a_i \in ((m_i - 1)\tau, m_i\tau]$ is such that $\bar{u}_\tau(a_i) = u_\tau^{m_i}$ with $m_0 = 0$, $m_n = N$, and $m_{i+1} > m_i$ for $i = 1, \ldots, N-1$. Then

$$\begin{aligned}
\|\bar{u}_\tau\|^2_{BV^2(0,T;V^*)} &= \sum_{i=1}^n \|u_\tau^{m_i} - u_\tau^{m_{i-1}}\|^2_{V^*} \\
&\le \sum_{i=1}^n \left((m_i - m_{i-1}) \sum_{k=m_{i-1}+1}^{m_i} \|u_\tau^k - u_\tau^{k-1}\|^2_{V^*} \right) \\
&\le \left(\sum_{i=1}^n (m_i - m_{i-1}) \right) \left(\sum_{i=1}^n \sum_{k=m_{i-1}+1}^{m_i} \|u_\tau^k - u_\tau^{k-1}\|^2_{V^*} \right) \\
&= N \sum_{k=1}^N \|u_\tau^k - u_\tau^{k-1}\|^2_{V^*} = T\tau \sum_{k=1}^N \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|^2_{V^*} \\
&\le CT\tau \sum_{k=1}^N \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|^2_{V}.
\end{aligned} \tag{5.11}$$

We now combine (4.15) and (5.11) to see that $\|\bar{u}_\tau\|^2_{BV^2(0,T;V^*)}$ is bounded, and we conclude from here that (5.7) holds.

For the proof of (5.9) we observe that

$$\|u'_\tau\|^2_{\mathcal{V}} = \tau \sum_{k=1}^N \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|^2_V,$$

and then we use (4.15). Finally, for the proof of (5.10) we note that

$$\|\bar{\xi}_\tau\|^2_{\mathcal{U}^*} = \sum_{k=1}^N \tau \|\xi_\tau^k\|^2_{U^*} \le \tau \sum_{k=1}^N c^2(1 + \|\iota u_\tau^k\|_U)^2 \le 2Tc^2 + 2c^2\tau \|\iota\|^2_{\mathcal{L}(V,U)} \sum_{k=1}^N \|u_\tau^k\|^2_V \le d_5.$$

This completes the proof of the lemma. ∎

We now have all the ingredients to provide the proof of Theorem 3.1.

Proof. First, it follows from Lemma 4.1 that, for $\tau > 0$ small enough, there exists a solution of Problem $\mathcal{P}_\tau$. We consider such a solution, and we use it to construct the functions u_τ, $\bar{u}_\tau$, and $\bar{\xi}_\tau$ defined by (5.1)–(5.3). We claim that there exists $u \in \mathcal{V}$ such that, passing to a subsequence again indexed by τ, the following convergences hold:

$$\bar{u}_\tau \to u \quad \text{weakly in } \mathcal{V}, \tag{5.12}$$

$$\bar{\iota}\bar{u}_\tau \to \bar{\iota}u \quad \text{strongly in } \mathcal{U}, \tag{5.13}$$

$$\bar{\xi}_\tau \to \xi \quad \text{weakly in } \mathcal{U}^*, \tag{5.14}$$

$$u_\tau \to u \quad \text{weakly in } \mathcal{V}, \tag{5.15}$$

$$u'_\tau \to u' \quad \text{weakly in } \mathcal{V}. \tag{5.16}$$

Indeed, it follows from Lemma 4.2, that, for $\tau > 0$ small enough, the estimates (5.6)–(5.10) hold. The convergence (5.12) to an element $u \in \mathcal{V}$ follows from the estimate (5.6), the continuous embedding $L^\infty(0,T;V) \subset \mathcal{V}$, and the reflexivity of the space $\mathcal{V}$. On the other hand, the estimate (5.7) combined with assumption $H(\iota)$ implies (5.13). The convergence (5.14) follows from the bound (5.10) and the reflexivity of the space $\mathcal{U}^*$. To prove (5.15) we recall that (5.9) implies that the sequence $\{u_\tau\}$ is bounded in $\mathcal{V}$, and therefore we can assume that there exists $u_1 \in \mathcal{V}$ such that $u_\tau \to u_1$ weakly in $\mathcal{V}$ as $\tau \to 0$. Thus, from (5.12) it follows that $\bar{u}_\tau - u_\tau \to u - u_1$ weakly in $\mathcal{V}$ as $\tau \to 0$. We also note that

$$\|\bar{u}_\tau - u_\tau\|^2_{\mathcal{V}} = \sum_{k=1}^{N} \int_{(k-1)\tau}^{k\tau} (k\tau - t)^2 \left\| \frac{u^k_\tau - u^{k-1}_\tau}{\tau} \right\|^2_V dt = \frac{\tau^2}{3} \|u'_\tau\|^2_{\mathcal{V}},$$

and therefore, from the bound (5.9) of $\{u'_\tau\}$, we deduce that $u = u_1$. We conclude from the above that (5.15) holds. The same bound (5.9) implies the convergence (5.16).

Next, we show that u is a solution of Problem $\mathcal{P}$. To this end we start with passing to the limit in the initial condition. Since the embedding $\{\, v \in \mathcal{V},\ v' \in \mathcal{V}\} \subset C(0,T;V)$ is continuous, from (5.15) and (5.16) it follows that $u_\tau \to u$ weakly in $C(0,T;V)$ as $\tau \to 0$. In particular we have

$$u_\tau(t) \to u(t) \quad \text{weakly in } V\,' \quad \text{as } \tau \to 0 \quad \text{for all } t \in [0,T]. \tag{5.17}$$

Hence, since $u_\tau(0) = u_0$ for all $\tau > 0$, we get $u(0) = u_0$. Now we pass to the limit in (5.5) as $\tau \to 0$. For $v \in \mathcal{V}$ we calculate

$$\langle \mathcal{A}u'_\tau, v - \bar{u}_\tau \rangle_{\mathcal{V}^* \times \mathcal{V}} = \langle \mathcal{A}u'_\tau, v \rangle_{\mathcal{V}^* \times \mathcal{V}} - \langle \mathcal{A}u'_\tau, u_\tau \rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle \mathcal{A}u'_\tau, u_\tau - \bar{u}_\tau \rangle_{\mathcal{V}^* \times \mathcal{V}}. \tag{5.18}$$

Since $\mathcal{A}$ is linear and continuous, it is also weakly continuous. So from (5.16) we have $\mathcal{A}u'_\tau \to \mathcal{A}u_\tau$ weakly in $\mathcal{V}^*$, i.e.,

$$\lim_{\tau \to 0} \langle \mathcal{A}u'_\tau, v \rangle_{\mathcal{V}^* \times \mathcal{V}} = \langle \mathcal{A}u', v \rangle_{\mathcal{V}^* \times \mathcal{V}}. \tag{5.19}$$

Moreover, we have

$$\begin{aligned} &\limsup_{\tau \to 0} \left(- \langle \mathcal{A}u'_\tau, u_\tau \rangle_{\mathcal{V}^* \times \mathcal{V}} \right) \\ &\quad = \limsup_{\tau \to 0} \left(\frac{1}{2} \langle Au_\tau(0), u_\tau(0) \rangle_{V^* \times V} - \frac{1}{2} \langle Au_\tau(T), u_\tau(T) \rangle_{V^* \times V} \right) \\ &\quad = \frac{1}{2} \langle Au(0), u(0) \rangle_{V^* \times V} - \liminf_{\tau \to 0} \frac{1}{2} \langle Au_\tau(T), u_\tau(T) \rangle_{V^* \times V}. \end{aligned} \tag{5.20}$$

It is easy to observe that the functional $V \ni v \to \langle Av, v\rangle_{V^*\times V}$ is continuous and convex and, therefore, that it is weakly lower semicontinuous. Thus, (5.17) yields

$$\langle Au(T), u(T)\rangle_{V^*\times V} \le \liminf_{\tau\to 0} \langle Au_\tau(T), u_\tau(T)\rangle_{V^*\times V}$$

and, combining this inequality with (5.20), we obtain

$$\limsup_{\tau\to 0} \left(- \langle \mathcal{A}u'_\tau, u_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}}\right) \le \frac{1}{2}\langle Au(0), u(0)\rangle_{V^*\times V} - \frac{1}{2}\langle Au(T), u(T)\rangle_{V^*\times V} = -\langle \mathcal{A}u', u\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{5.21}$$

Next, an elementary calculus shows that

$$\begin{aligned}
\langle \mathcal{A}u'_\tau, u_\tau - \bar{u}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \sum_{k=1}^{N} \int_{\tau(k-1)}^{\tau k} \langle Au'_\tau(t), u_\tau(t) - \bar{u}_\tau(t)\rangle_{V^*\times V}\, dt \\
&= \sum_{k=1}^{N} \int_{\tau(k-1)}^{\tau k} \left\langle A\frac{1}{\tau}\left(u_\tau^k - u_\tau^{k-1}\right), \left(\frac{t}{\tau} - k\right)\left(u_\tau^k - u_\tau^{k-1}\right)\right\rangle_{V^*\times V} dt \\
&= \sum_{k=1}^{N} \frac{1}{\tau}\left\langle A\left(u_\tau^k - u_\tau^{k-1}\right), u_\tau^k - u_\tau^{k-1}\right\rangle_{V^*\times V} \int_{\tau(k-1)}^{\tau k}\left(\frac{t}{\tau} - k\right) dt \\
&= -\sum_{k=1}^{N} \frac{1}{2}\left\langle A\left(u_\tau^k - u_\tau^{k-1}\right), u_\tau^k - u_\tau^{k-1}\right\rangle_{V^*\times V} \le 0.
\end{aligned}$$

Combining this inequality with (5.18), (5.19), and (5.21) we obtain that

$$\limsup_{\tau\to 0} \langle \mathcal{A}u'_\tau, v - \bar{u}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} \le \langle \mathcal{A}u', v - u\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{5.22}$$

We now estimate

$$\limsup_{\tau\to 0} \langle \mathcal{B}\bar{u}_\tau, v - \bar{u}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} \le \limsup_{\tau\to 0} \langle \mathcal{B}\bar{u}_\tau, v\rangle_{\mathcal{V}^*\times\mathcal{V}} - \liminf_{\tau\to 0} \langle \mathcal{B}\bar{u}_\tau, \bar{u}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{5.23}$$

Since the operator $\mathcal{B}$ is linear and continuous, it is also weakly continuous. So from (5.16) we have $\mathcal{B}u'_\tau \to \mathcal{B}u_\tau$ weakly in $\mathcal{V}^*$, i.e.,

$$\limsup_{\tau\to 0} \langle \mathcal{B}\bar{u}_\tau, v\rangle_{\mathcal{V}^*\times\mathcal{V}} = \lim_{\tau\to 0} \langle \mathcal{B}\bar{u}_\tau, v\rangle_{\mathcal{V}^*\times\mathcal{V}} = \langle \mathcal{B}u, v\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{5.24}$$

Using now a lower semicontinuity argument, it follows that

$$\liminf_{\tau\to 0} \langle \mathcal{B}\bar{u}_\tau, \bar{u}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} \ge \langle \mathcal{B}u, u\rangle_{\mathcal{V}^*\times\mathcal{V}}$$

and, combining this inequality with (5.23) and (5.24), we obtain that

$$\limsup_{\tau\to 0} \langle \mathcal{B}\bar{u}_\tau, v - \bar{u}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} \le \langle \mathcal{B}u, v - u\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{5.25}$$

Moreover, from (4.2) and (5.12) we have

$$\langle f_\tau, v - \bar{u}_\tau\rangle_{\mathcal{V}^*\times\mathcal{V}} \to \langle f, v - u\rangle_{\mathcal{V}^*\times\mathcal{V}} \quad \text{as } \tau \to 0 \tag{5.26}$$

and, using (5.13) and (5.14), we see that

$$\langle \bar{\xi}_\tau, \bar{\iota}(v - \bar{u}_\tau)\rangle_{\mathcal{U}^*\times\mathcal{U}} \to \langle \xi, \bar{\iota}(v-u)\rangle_{\mathcal{U}^*\times\mathcal{U}} \quad \text{as } \tau \to 0. \tag{5.27}$$

Finally, we define the functional $\Psi : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ by

$$\Psi(v) = \int_0^T \Phi(v(t))\, dt \qquad \text{for all } v \in \mathcal{V}.$$

We show in what follows that Ψ is lower semicontinuous. Indeed, let $u_n \to u$ strongly in $\mathcal{V}$. By Lemma 2.5 in [27] there exist $k_1, k_2 \in \mathbb{R}$ such that for $v \in V$ we have $\Phi(v) \geq k_1\|v\|_V + k_2$, which implies that

$$\int_0^T \Phi(u_n(t))\, dt \geq k_2 T + k_1 \int_0^T \|u_n(t)\|_V\, dt \geq k_2 T - |k_1|\sqrt{T}\|u_n\|_{\mathcal{V}} \geq k_0, \tag{5.28}$$

where k_0 is a positive constant which does not depend on n.

Consider now a convergent subsequence of $\Psi(u_n)$, i.e., $\Psi(u_n) \to M$ as $n \to \infty$. Then, for a subsequence, we have $u_{n_k}(t) \to u(t)$ strongly in V for a.e. $t \in (0,T)$. From the lower semicontinuity of Φ we have

$$\Phi(u(t)) \leq \liminf_{n\to\infty} \Phi(u_{n_k}(t)) \quad \text{a.e. } t \in (0,T).$$

Inequality (5.28) allows us to use the Fatou lemma. As a result we obtain

$$\int_0^T \Phi(u(t))\, dt \leq \int_0^T \liminf_{n_k\to\infty} \Phi(u_{n_k}(t))\, dt \leq \liminf_{n_k\to\infty} \int_0^T \Phi(u_{n_k}(t))\, dt = M,$$

which shows that Ψ is a lower semicontinuous function. Moreover, since Φ is convex, then so is Ψ, and, as a consequence, it is weakly sequentially lower semicontinuous. From (5.12) we have $\Psi(u) \leq \liminf_{\tau\to 0}\Psi(\bar{u}_\tau)$, and

$$\limsup_{\tau\to 0} \int_0^T (\Phi(v(t)) - \Phi(\bar{u}_\tau(t)))dt \leq \int_0^T (\Phi(v(t)) - \Phi(u(t)))dt. \tag{5.29}$$

We now use (5.22), (5.25)–(5.27), (5.29), and inequality (5.5) to see that (u, ξ) satisfies (3.1). Moreover, since $\bar{\xi}_\tau(t) \in \partial_{Cl} J(\bar{\iota}\bar{u}_\tau(t))$ for a.e. $t \in (0,T)$, from (5.13), (5.14), and the convergence theorem of Aubin and Cellina (see [1], for instance) we have

$$\eta(t) \in \partial_{Cl} J(\bar{\iota} u(t)) \quad \text{for a.e. } t \in (0,T).$$

We deduce from the above that u is a solution of Problem $\mathcal{P}$. The regularity $u \in H^1(0,T;V)$ follows from (5.15)–(5.16), which concludes the existence part of the theorem.

To prove the uniqueness part we assume in what follows that u_1 and u_2 are two solutions of Problem $\mathcal{P}$, and we let $t \in [0,T]$. We write inequality (3.1) for $u = u_1$ with

$$v(s) = \begin{cases} u_2(t) & \text{if } s \in [0,t], \\ u_1(t) & \text{if } s \in [t,T] \end{cases}$$

and for $u = u_2$ with

$$v(s) = \begin{cases} u_1(t) & \text{if } s \in [0,t], \\ u_2(t) & \text{if } s \in [t,T]; \end{cases}$$

then we add the resulting inequalities. In this way we obtain

$$\begin{aligned} (5.30) \qquad & \int_0^t \langle A(u_1'(s) - u_2'(s)), u_1(s) - u_2(s)\rangle_{V^*\times V}\, ds \\ & + \int_0^t \langle B(u_1(s) - u_2(s)), u_1(s) - u_2(s)\rangle_{V^*\times V}\, ds \\ & + \int_0^t \langle \xi_1(s) - \xi_2(s), \iota u_1(s) - \iota u_2(s)\rangle_{U^*\times U}\, ds \le 0, \end{aligned}$$

where $\xi_i(s) \in \partial_{Cl} J(\iota u_i(s))$ for a.e. $s \in (0,t)$, $i = 1, 2$. Using now $H(A)$, $H(B)$, $H(J)$, and (5.30) yields

$$\begin{aligned} (5.31) \qquad & \alpha \|u_1(t) - u_2(t)\|_V^2 \le \langle A(u_1(t) - u_2(t)), u_1(t) - u_2(t)\rangle_{V^*\times V} \\ & + \int_0^t \beta \|u_1(s) - u_2(s)\|_V^2\, ds \le \|\iota\|^2_{\mathcal{L}(V,U)} \int_0^t m\|u_1(s) - u_2(s)\|_V^2\, ds \\ & + \langle A(u_1(0) - u_2(0)), u_1(0) - u_2(0)\rangle_{V^*\times V}. \end{aligned}$$

Recall also that $u_1(0) = u_2(0) = u_0$, and therefore (5.31) implies that

$$\alpha \|u_1(t) - u_2(t)\|_V^2 \le \|\iota\|^2_{\mathcal{L}(V,U)} \int_0^t m\|u_1(s) - u_2(s)\|_V^2\, ds.$$

We now use the Gronwall lemma to conclude that $u_1(t) = u_2(t)$ for all $t \in (0,T)$, which completes the proof. ☐

6. A frictionless contact problem. A large number of quasi-static contact problems with viscoelastic materials lead to evolutionary variational-hemivariational inequalities of the form (3.1), (3.2) in which the unknown is the displacement field. For a variety of such inequalities, Theorem 3.1 can be applied. In this section we illustrate this point on a viscoelastic frictionless contact problem.

The physical setting is the following. A viscoelastic body occupies, in its reference configuration, a regular domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with boundary $\partial\Omega$. The boundary is partitioned into three disjoint measurable parts Γ_1, Γ_2, and Γ_3 such that the measure of Γ_1, denoted by $m(\Gamma_1)$, is positive. The body is clamped on Γ_1, and so the displacement field vanishes there. Time-dependent surface tractions of density $\boldsymbol{f}_2$ act on Γ_2, and time-dependent volume forces of density $\boldsymbol{f}_0$ act on Ω. The body is in frictionless contact on Γ_3 with an obstacle, the so-called foundation. The foundation is made of a perfectly rigid material, covered by a layer of deformable material of thickness $g > 0$. Therefore, we model the contact with a normal compliance condition associated to a unilateral contact condition. The process is quasi-static, and the time interval of interest is $[0,T]$, where $T > 0$. Then the mathematical model of the contact problem (that we state here and explain below in this section) is the following.

Problem $\mathcal{P}_{\mathcal{M}}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega \times (0,T), \tag{6.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega \times (0,T), \tag{6.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1 \times (0,T), \tag{6.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{6.4}$$

$$\left.\begin{aligned} &\sigma_\nu = \sigma_\nu^1 + \sigma_\nu^2, \\ &-\sigma_\nu^1 \in \partial_{Cl} j(u_\nu), \\ &u_\nu \le g, \ \ \sigma_\nu^2 \le 0, \ \ \sigma_\nu^2(u_\nu - g) = 0 \end{aligned}\right\} \qquad \text{on } \Gamma_3 \times (0,T), \tag{6.5}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \qquad \text{on } \Gamma_3 \times (0,T), \tag{6.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{0} \qquad \text{in } \Omega. \tag{6.7}$$

Note that $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. Recall that the canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \text{for all } \ \boldsymbol{u} = (u_i),\, \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \quad \text{for all } \ \boldsymbol{\sigma} = (\sigma_{ij}),\, \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

respectively. Here and below, the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used.

In (6.1)–(6.7) we use the notation $\boldsymbol{u} = (u_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$, and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ for the displacement vector, the stress tensor, and the linearized strain tensor, respectively, and $\boldsymbol{\nu} = (\nu_i)$ for the outward unit normal at $\partial\Omega$. Recall that $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$, where an index following a comma indicates a partial derivative with respect to the corresponding component of the spatial variable, denoted by $\boldsymbol{x} = (x_i)$. Moreover, we use the notation v_ν and $\boldsymbol{v}_\tau$ for the normal and tangential components of a vector field $\boldsymbol{v}$ on $\partial\Omega$, i.e., $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$. In addition, σ_ν and $\boldsymbol{\sigma}_\tau$ represent the normal and tangential components of the stress field $\boldsymbol{\sigma}$ and are defined by $\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$, respectively.

We now present a short description of the equations and conditions in Problem $\mathcal{P}_{\mathcal{M}}$ in which, for simplicity, we do not indicate explicitly the dependence of the variables on $\boldsymbol{x}$. We refer the reader to [13, 18, 25, 32, 34] for more details on mathematical models in contact mechanics, including additional explanation related to our comments below.

First, (6.1) is the Kelvin–Voigt viscoelastic constitutive law, used in the literature to model the behavior of real bodies like metals, rubbers, and various polymers. Equation (6.2) represents the equation of equilibrium, and we use it since we assume that the process is quasi-static. Next, conditions (6.3) and (6.4) are the displacement and the traction boundary condition, respectively, while (6.6) and (6.7) represent the frictionless condition and the initial condition, respectively.

We turn now to the contact condition (6.5) in which our interest lies and which represents the main trait of novelty of the mechanical model we consider in this paper. This condition is obtained by assuming that the normal stress on the contact surface, denoted by σ_ν, has an additive decomposition in two parts, σ_ν^1 and σ_ν^2. Part σ_ν^1 describes the deformability of the obstacle, and therefore it follows a normal compliance condition governed by the subdifferential of a nonconvex potential j. Part σ_ν^2 describes the rigidity of the obstacle, and therefore it satisfies the Signorini unilateral contact condition. Recall that condition (6.5) models the contact with a foundation which is made by a rigid body covered by a layer made of elastic material, say asperities. It shows that the penetration is restricted, since $u_\nu \le g$, where g represents the thickness of the elastic layer. Also, when there is penetration, as far as the normal displacement does not reach the bound g, the contact is described with a nonmonotone normal compliance condition, since in this case $-\sigma_\nu \in \partial j_\nu(u_\nu)$. Due to the nonmonotonicity of ∂j_ν, the condition allows one to describe the hardening or the softening phenomenon of the foundation. Various examples and mechanical interpretation associated with the nonmonotone normal compliance condition can be found in [25].

In the study of Problem $\mathcal{P}_\mathcal{M}$ we use standard notation for Lebesgue and Sobolev spaces. For all $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$ we denote by $\gamma\boldsymbol{v}$ the trace of $\boldsymbol{v}$ on $\partial\Omega$ and, recall, we use the notation v_ν and $\boldsymbol{v}_\tau$ for its normal and tangential traces. In addition, we introduce spaces V and $\mathcal{H}$ defined by

$$V = \{\ \boldsymbol{v} = (v_i) \in H^1(\Omega;\mathbb{R}^d) \mid \gamma\boldsymbol{v} = \boldsymbol{0} \ \ \text{a.e. on } \Gamma_1\ \},$$
$$\mathcal{H} = \{\ \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega;\mathbb{R}^{d\times d}) \mid \tau_{ij} = \tau_{ji},\ 1 \le i,j \le d\ \}.$$

The space $\mathcal{H}$ is a real Hilbert space with the canonical inner product given by

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_\mathcal{H} = \int_\Omega \sigma_{ij}(\boldsymbol{x})\,\tau_{ij}(\boldsymbol{x})\,dx$$

and the associated norm $\|\cdot\|_\mathcal{H}$. Since $m(\Gamma_1) > 0$, it is well known that V is a real Hilbert space with the inner product

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_\mathcal{H}, \quad \boldsymbol{u},\boldsymbol{v} \in V, \tag{6.8}$$

and the associated norm $\|\cdot\|_V$. The duality pairing between V and V^* is denoted by $\langle\cdot,\cdot\rangle_{V^*\times V}$. Let $\varepsilon \in (0,\frac{1}{2})$ and $Z = H^{1-\varepsilon}(\Omega;\mathbb{R}^d)$. We also define the embedding $i_1 : V \to Z$, the trace operator $\gamma_1 : Z \to H^{\frac{1}{2}-\varepsilon}(\Gamma_3;\mathbb{R}^d)$, and the embedding $i_3 : H^{\frac{1}{2}-\varepsilon}(\Gamma_3;\mathbb{R}^d) \to L^2(\Gamma_3;\mathbb{R}^d)$, and we consider the trace operator $\gamma : V \to L^2(\Gamma_3;\mathbb{R}^d)$ defined by $\gamma = i_2 \circ \gamma_1 \circ i_1$. By the Sobolev trace theorem there exists a constant c_0 depending only on the domain Ω, Γ_1, and Γ_3 such that

$$\|\gamma\boldsymbol{v}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le c_0\|\boldsymbol{v}\|_V \quad \text{for all}\,\boldsymbol{v} \in V. \tag{6.9}$$

Let us denote $U = L^2(\Gamma_3)$ and define the operators $\nu : L^2(\Gamma_3;\mathbb{R}^d) \to U$ and $\iota : V \to U$ by equalities $\nu\boldsymbol{v} = v_\nu$ for all $\boldsymbol{v} \in L^2(\Gamma_3;\mathbb{R}^d)$ and $\iota : V \to U$ by $\iota = \nu \circ \gamma$. In addition, we consider the spaces $\mathcal{V}$, $\mathcal{U}$, and $\mathcal{W}$ defined at the beginning of section 2, with the previous notation for V and U.

We now consider the following assumptions on the data of Problem $\mathcal{P}_\mathcal{M}$.

$H(\mathcal{C})$. The viscosity tensor $\mathcal{C} = (\mathcal{C}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) } \mathcal{C}_{ijkl} \in L^\infty(\Omega), \quad 1 \le i,j,k,l \le d; \\ \text{(b) } \mathcal{C}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{C}\boldsymbol{\tau} \quad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d \text{ a.e. in } \Omega; \\ \text{(c) } \mathcal{C}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge \alpha |\boldsymbol{\tau}|^2 \quad \text{for all } \boldsymbol{\tau} \in \mathbb{S}^d \text{ a.e. in } \Omega \text{ with } \alpha > 0. \end{cases}$$

$H(\mathcal{G})$. The elasticity tensor $\mathcal{G} = (\mathcal{G}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) } \mathcal{G}_{ijkl} \in L^\infty(\Omega), \ 1 \le i,j,k,l \le d; \\ \text{(b) } \mathcal{G}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{G}\boldsymbol{\tau} \quad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d \text{ a.e. in } \Omega; \\ \text{(c) } \mathcal{G}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge \beta |\boldsymbol{\tau}|^2 \quad \text{for all } \boldsymbol{\tau} \in \mathbb{S}^d \text{ a.e. in } \Omega \text{ with } \beta > 0. \end{cases}$$

$H(j)$. The normal compliance function $j : \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) } j \text{ is locally Lipschitz;} \\ \text{(b) } \partial_{Cl} j \text{ satisfies the growth condition } |\eta| \le c_1(1 + |s|) \\ \qquad \text{for all } \eta \in \partial_{Cl} j(s),\ s \in \mathbb{R} \text{ with } d > 0; \\ \text{(c) } (\eta_1 - \eta_2)(s_1 - s_2) \ge -c_2 |s_1 - s_2|^2 \text{ for all } \eta_i \in \partial_{Cl} j(s_i), \\ \qquad s_i \in \mathbb{R},\ i = 1, 2 \text{ with } c_2 > 0. \end{cases}$$

$H(f)$. The densities of forces and traction satisfy

$$\boldsymbol{f}_0 \in H^1(0,T; L^2(\Omega;\mathbb{R}^d)), \quad \boldsymbol{f}_2 \in H^1(0,T; L^2(\Gamma_2;\mathbb{R}^d));$$
$$\boldsymbol{f}_0(0) \in V, \quad \boldsymbol{f}_2(0) = \boldsymbol{0}.$$

In order to derive a variational formulation of Problem $\mathcal{P}_\mathcal{M}$, we define operators $A, B : V \to V^*$ by equalities

$$\langle A\boldsymbol{u}, \boldsymbol{v} \rangle_{V^* \times V} = (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V, \tag{6.10}$$
$$\langle B\boldsymbol{u}, \boldsymbol{v} \rangle_{V^* \times V} = (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V. \tag{6.11}$$

We also define the functions $J : L^2(\Gamma_3) \to \mathbb{R}$ and $\boldsymbol{f} : (0,T) \to V^*$ by equalities

$$J(w) = \int_{\Gamma_3} j(w)\, d\Gamma \quad \text{for } w \in L^2(\Gamma_3), \tag{6.12}$$

$$\langle \boldsymbol{f}(t), \boldsymbol{v} \rangle_{V^* \times V} = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, d\Gamma \quad \text{for all } \boldsymbol{v} \in V \text{ and all } t \in [0,T]. \tag{6.13}$$

Finally, let $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ be an indicator function of the set

$$K = \{\boldsymbol{v} \in V,\ \boldsymbol{v}_\nu \le g \text{ a.e. on } \Gamma_3\},$$

i.e.,

$$\Phi(\boldsymbol{v}) = \begin{cases} 0 & \text{if } \boldsymbol{v} \in K, \\ +\infty & \text{otherwise.} \end{cases} \tag{6.14}$$

Assume now that $(\boldsymbol{u}, \boldsymbol{\sigma})$ represent a regular solution of the contact problem (6.1)–(6.7), and let $\boldsymbol{v} \in \mathcal{V}$. Then, using a standard procedure based on the Green formula and notation (6.10), (6.11), (6.13), we deduce that

$$\langle A\boldsymbol{u}'(t) + B\boldsymbol{u}(t) - \boldsymbol{f}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t)\rangle_{V^*\times V} - \int_{\Gamma_3} \sigma_\nu^1(t)(v_\nu(t) - u_\nu(t))\, d\Gamma$$
$$= \int_{\Gamma_3} \sigma_\nu^2(t)(v_\nu(t) - u_\nu(t))\, d\Gamma \qquad \text{a.e. } t \in (0,T). \tag{6.15}$$

On the other hand, (6.6) and notation (6.14) imply that

$$\int_{\Gamma_3} \sigma_\nu^2(v_\nu(t) - u_\nu(t))\, d\Gamma \geq \Phi(\boldsymbol{u}(t)) - \Phi(\boldsymbol{v}(t)) \qquad \text{a.e. } t \in (0,T). \tag{6.16}$$

Next, we denote

$$-\sigma_\nu(t) = \xi(t) \tag{6.17}$$

and use again (6.6) and notation (6.12) to deduce that

$$\xi(t) \in \partial_{Cl} J(\iota \boldsymbol{u}(t)) \qquad \text{a.e. } t \in (0,T). \tag{6.18}$$

It follows now from (6.15)–(6.17) that

$$\langle A\boldsymbol{u}'(t) + B\boldsymbol{u}(t) + \iota^*\xi(t) - \boldsymbol{f}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t)\rangle_{V^*\times V}$$
$$+\Phi(\boldsymbol{v}(t)) - \Phi(\boldsymbol{u}(t)) \geq 0 \qquad \text{a.e. } t \in (0,T).$$

We now integrate (6.19) on $[0,T]$ and combine the resulting inequality with inclusion (6.18) and the initial condition (6.7) to obtain the following variational formulation of Problem $\mathcal{P}_{\mathcal{M}}$ in term of the displacements.

Problem ${\mathcal{P}_{\mathcal{M}}}^V$. *Find a displacement field $\boldsymbol{u} \in \mathcal{W}$ such that $\boldsymbol{u}(0) = \boldsymbol{0}$ and, moreover,*

$$\int_0^T \langle A\boldsymbol{u}'(t) + B\boldsymbol{u}(t) + \iota^*\xi(t) - \boldsymbol{f}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t)\rangle_{V^*\times V}\, dt$$
$$+ \int_0^T \big(\Phi(\boldsymbol{v}(t)) - \Phi(\boldsymbol{u}(t))\big)\, dt \geq 0 \quad \textit{for all } \boldsymbol{v} \in \mathcal{V}$$

with

$$\xi(t) \in \partial_{Cl} J(\iota \boldsymbol{u}(t)) \quad \textit{for a.e. } t \in (0,T)$$

The main result of this section states the unique solvability of Problem $\mathcal{P}_{\mathcal{M}}$ and is formulated as follows.

Theorem 6.1. *Assume that $H(\mathcal{C})$, $H(\mathcal{G})$, $H(j)$, and $H(f)$ hold and, moreover, that*

$$\beta > c_2. \tag{6.19}$$

Then Problem $\mathcal{P}_{\mathcal{M}}^{\mathcal{V}}$ has a unique solution $\boldsymbol{u} \in H^1(0,T;V)$.

Proof. We apply Theorem 3.1. To this end we note that assumptions $H(\mathcal{C})$ and $H(\mathcal{G})$ imply that the operators A and B defined by (6.10) and (6.11) satisfy the hypotheses $H(A)$ and $H(B)$, respectively. Also, assumption $H(j)$ implies that the functional J defined by (6.12) satisfies the assumption $H(J)$. The functional Φ defined by (6.14) satisfies the hypothesis $H(\Phi)$. Since $\boldsymbol{u}_0 = \mathbf{0}_V$, we have $\boldsymbol{u}_0 \in K = dom(\Phi)$. Moreover, it is easy to see that $\mathbf{0}_V \in \partial_{Conv}\Phi(\boldsymbol{u}_0)$ and, by the basic properties of the Clarke subdifferential, that the set $\partial_{Cl}J(\iota\boldsymbol{u}_0)$ is nonempty. By $H(f)$ and (6.13) we obtain $\boldsymbol{f}(0) \in V$, and by $H(\mathcal{C})$ and (6.11) we have $B\boldsymbol{u}_0 = \mathbf{0}_V \in V$. This implies that condition $H(0)$ is satisfied with some $\boldsymbol{\xi}_0 \in \partial_{Cl}J(\iota\boldsymbol{u}_0)$ and $\boldsymbol{\eta}_0 = \mathbf{0}_V$. Finally, (6.19) implies that condition $H(s)$ is satisfied, too.

Now we will show that the operator ι satisfies $H(\iota)$. To this end we consider a bounded sequence $\{\boldsymbol{v}_n\} \subset M^{2,2}(0,T;V,V^*)$. We show that for a subsequence, still denoted by $\{\boldsymbol{v}_n\}$, we have

$$\bar{\iota}\boldsymbol{v}_n \to \boldsymbol{u} \text{ strongly in } \mathcal{U} \text{ as } n \to \infty \tag{6.20}$$

with $\boldsymbol{u} \in \mathcal{U}$. To this end we note that, since the embedding $i_1 : V \to Z$ is compact and the embedding $Z \subset V^*$ is continuous, from Proposition 2.8, it follows that for a subsequence, still denoted by $\{\boldsymbol{u}_n\}$, we have

$$\int_0^T \|i_1\boldsymbol{u}_n(t) - \boldsymbol{z}(t)\|_Z^2\,dt \to 0 \text{ with } n \to \infty, \tag{6.21}$$

where $\boldsymbol{z} \in L^2(0,T;Z)$. Let $\boldsymbol{u} \in \mathcal{U}$ be given by $\boldsymbol{u} = (\nu \circ i_2 \circ \gamma_1)\boldsymbol{z}$. We have

$$\begin{aligned}\|\bar{\iota}\boldsymbol{v}_n - \boldsymbol{u}\|_{\mathcal{U}}^2 &= \int_0^T \|\iota\boldsymbol{v}_n(t) - \boldsymbol{u}(t)\|_U^2\,dx \\ &\le \|\nu \circ i_2 \circ \gamma_1\|_{\mathcal{L}(Z;U)}^2 \int_0^T \|i_1\boldsymbol{u}_n(t) - \boldsymbol{z}(t)\|_Z^2\,dx.\end{aligned}$$

We combine this inequality with (6.21) to see that (6.20) holds. We conclude from here that condition $H(\iota)$ is satisfied. Theorem 6.1 is now a direct consequence of Theorem 3.1. ∎

A couple of functions $(\boldsymbol{u}, \boldsymbol{\sigma})$ such that $\boldsymbol{u}$ is a solution of Problem $\mathcal{P}_{\mathcal{M}}^{\mathcal{V}}$ and $\boldsymbol{\sigma}$ is given by (6.1) is called a weak solution to Problem $\mathcal{P}_{\mathcal{M}}$. We conclude that under the assumptions of Theorem 6.1 contact problem $\mathcal{P}_{\mathcal{M}}$ has a unique weak solution. Moreover, the solution has the regularity $\boldsymbol{u} \in H^1(0,T;V)$ and $\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H})$.

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