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# Domination and Compensation in Finite Dimension Dynamical Systems

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## Résumé

This work concerns the analysis of a class of linear dynamical systems. We study the possibility of comparing input operators, with respect to the output one, and we give characterization results. Various situations are examined, applications and illustrative examples are presented.

Under convenient hypothesis, we also show how to find the optimal control ensuring the compensation of a disturbance in the finite time or asymptotic cases. The relationship with the notions of controllability, stability and stabilizability is examined. Here also, applications and examples illustrating different results and situations are given.

**Keywords :** Dynamical systems, observation, control, domination, remediability, disturbance

## 1 Introduction and problem statement

In this work, we consider a class of finite dimension dynamical systems described by a linear state equation as follows :

$$(S) \begin{cases} \dot{z}(t) = Az(t) + B_1u_1(t) + B_2u_2(t) ; 0 < t < T \\ z(0) = z_0 \in \mathbb{R}^n \end{cases} \quad (1)$$

where  $A \in M_n(\mathbb{R})$ ,  $B_1 \in M_{n,p}(\mathbb{R})$ ,  $B_2 \in M_{n,m}(\mathbb{R})$ ,  $u_1 \in L^2(0, T; \mathbb{R}^p)$  and  $u_2 \in L^2(0, T; \mathbb{R}^m)$ .

The system (S) is augmented by the output equation :

$$y(t) = Cz(t) ; 0 < t < T \quad (2)$$

where  $C \in M_{q,n}(\mathbb{R})$ . The state of system (S) at time  $t$ , is given by :

$$z(t) = e^{At}z_0 + H_1(t)u_1 + H_2(t)u_2 \quad (3)$$

where for  $t \in ]0, T]$ ,  $H_1(t)$  and  $H_2(t)$  are the operators defined by

$$\begin{aligned} H_1(t) : L^2(0, t; \mathbb{R}^p) &\longrightarrow \mathbb{R}^n \\ u_1 &\longrightarrow \int_0^t e^{A(t-s)} B_1 u_1(s) ds \end{aligned} \quad (4)$$

and

$$\begin{aligned} H_2(t) : L^2(0, t; \mathbb{R}^m) &\longrightarrow \mathbb{R}^n \\ u_2 &\longrightarrow \int_0^t e^{A(t-s)} B_2 u_2(s) ds \end{aligned} \quad (5)$$

For  $i = 1, 2$ , we note  $H_i = H_i(T)$ . Then

$$y(T) = Ce^{AT}z_0 + CH_1u_1 + CH_2u_2 \quad (6)$$

The system (S) is excited by two input terms  $B_1u_1$  and  $B_2u_2$  where one, the second for example, is considered as an intentional or accidental disturbance. The other term  $B_1u_1$  is introduced in order to compensate [1,2] at the final time  $T$ , the effect of the disturbance by bringing back the observation to the normal situation which is  $Ce^{AT}z_0$ . That is to say : for any  $u_2 \in L^2(0, T; \mathbb{R}^m)$ , there exists a control  $u_1 \in L^2(0, T; \mathbb{R}^p)$  such that :

$$Ce^{A(T)}z_0 + \int_0^T Ce^{A(T-s)} B_1 u_1(s) ds + \int_0^T Ce^{A(T-s)} B_2 u_2(s) ds = Ce^{A(T)}z_0$$

or equivalently

$$CH_1u_1 + CH_2u_2 = 0$$

This leads to the notion of domination which consists to study the possibility of comparing the input operators  $B_1$  and  $B_2$ , with respect to the output one  $C$ .

The domination notion is introduced and studied separately for controlled and observed distributed systems [3]. In this paper, we examine the problem of domination in connection with the compensation one.

First, we consider the finite time case. We define and we characterize the notion of domination. Sufficient conditions, applications and illustrative examples are also given. The minimum energy problem [7,8] is examined using Hilbert Uniqueness Method, such a problem can be studied as a general optimal control one. The obtained results are extended to the asymptotic case and various situations are examined. The relationship with the notions of stability and stabilizability [4,5,6] are equally studied.

## 2 Finite time $C$ -domination

### 2.1 Definitions and characterizations

We define hereafter the notion of  $C$ -domination.

#### Definition 2.1

We say that  $B_1$  dominates  $B_2$  on  $[0, T]$  with respect to  $C$  (or  $B_1$   $C$ -dominates  $B_2$  on  $[0, T]$ ), if for any  $u_2 \in L^2(0, T; \mathbb{R}^m)$ , there exists a control  $u_1 \in L^2(0, T; \mathbb{R}^p)$  such that :

$$CH_1u_1 + CH_2u_2 = 0$$

In this case, and for  $C$  and  $T$  fixed, one can note  $B_2 \leq B_1$

We have the following characterization result.

**Proposition 2.2** *The following properties are equivalent :*

- i)  $B_1$  dominates  $B_2$  on  $[0, T]$  with respect to  $C$ .
- ii)  $Im(CH_2) \subset Im(CH_1)$ .
- iii)  $Ker(H_1^*C^*) \subset Ker(H_2^*C^*)$ .

iv)  $\exists \gamma > 0$  such that for any  $\theta \in \mathbb{R}^q$ , we have

$$\|B_2^* e^{A^*(T-\cdot)} C^* \theta\|_{L^2(0,T;\mathbb{R}^m)} \leq \gamma \|B_1^* e^{A^*(T-\cdot)} C^* \theta\|_{L^2(0,T;\mathbb{R}^p)} \quad (7)$$

Proof : Derive from the definition, the fact that

$$\text{Ker}(H_i^* C^*) = \text{Ker}(B_i^* e^{A^*(T-\cdot)} C^*), \quad \text{for } i = 1, 2$$

and also the following well known result.

**Lemma 2.3**

Let  $X$ ,  $Y$  and  $Z$  be Banach reflexive spaces,  $P \in \mathcal{L}(X, Z)$  and  $Q \in \mathcal{L}(Y, Z)$ . We have

$$\text{Im}(P) \subset \text{Im}(Q)$$

if and only if

$$\exists \gamma > 0 \text{ such that for any } z^* \in Z', \text{ we have } \|P^* z^*\|_{X'} \leq \gamma \|Q^* z^*\|_{Y'}$$

Concerning the relationship with the controllability notion, we have the following result.

**Proposition 2.4**

i) If the system

$$(S_1) \begin{cases} \dot{z}(t) = Az(t) + B_1 u_1(t); & 0 < t < T \\ z(0) = z_0 \in \mathbb{R}^n \end{cases} \quad (8)$$

is controllable on  $[0, T]$ , then  $B_1$  dominates any operator  $B_2$  on  $[0, T]$ , with respect<sup>1</sup> to  $C$ .

ii) The converse is not true.

Proof :

i) Obviously,  $(S_1)$  is controllable on  $[0, T] \iff \text{Im}H_1 = \mathbb{R}^n$ , then

$$\text{Im}(CH_1) = \text{Im}(C)$$

and hence

---

<sup>1</sup>In fact, this is true for any output operator  $C$ .

$$Im(CH_2) \subset Im(CH_1)$$

Consequently  $B_1$  dominates  $B_2$  on  $[0, T]$ , with respect to  $C$ .

ii) Counter example : We consider the case where  $n = 2, p = q = 1$  and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad C = (1 \ 0)$$

we have

$$B_2^* e^{A^*(T-s)} C^* \theta = e^{(T-s)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \theta = \begin{pmatrix} e^{(T-s)} \theta \\ 0 \end{pmatrix}$$

and

$$B_1^* e^{A^*(T-s)} C^* \theta = (1 \ 0) \begin{pmatrix} e^{(T-s)} \theta \\ 0 \end{pmatrix} = e^{(T-s)} \theta$$

then

$$\| B_2^* e^{A^*(T-\cdot)} C^* \theta \|_{L^2(0,T;\mathbb{R}^2)} = \| B_1^* e^{A^*(T-\cdot)} C^* \theta \|_{L^2(0,T;\mathbb{R})}$$

The inequality (7) is then true for  $\gamma = 1$ . Hence  $B_1$   $C$ -dominates  $B_2$ , but

$$rank \begin{pmatrix} B_1 & AB_1 \end{pmatrix} = rank \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 1 < 2$$

Consequently  $(S_1)$  is not controllable on  $[0, T]$ . □

We give hereafter a sufficient condition ensuring such a domination.

**Proposition 2.5** *If*

$$rank \begin{pmatrix} CB_1 & CAB_1 & \dots & CA^{n-1}B_1 \end{pmatrix} = q \tag{9}$$

*then  $B_1$   $C$ -dominates any operator  $B_2$  on  $[0, T]$ .*

Proof : Using Caylay-Hamilton theorem, we have

$$rank \begin{pmatrix} CB_1 & CAB_1 & \dots & CA^{n-1}B_1 \end{pmatrix} = q$$

$$\iff \begin{pmatrix} B_1^* C^* \\ B_1^* A^* C^* \\ \vdots \\ B_1^* (A^*)^{n-1} C^* \end{pmatrix}_{(np,q)} y = 0 ; \forall y \in \mathbb{R}^q \implies y = 0$$

$$\iff Ker(H_1)^* C^* = \{0\}$$

Hence, if  $Ker [(H_1)^*C^*] = \{0\}$ , then  $Ker [(H_1)^*C^*] \subset Ker [(H_2)^*C^*]$  and then,  $B_1$  dominates  $B_2$  on  $[0, T]$  with respect to  $C$ .  $\square$

**Remark 2.6**

*i) One can have*

$$\text{rank} ( CB_1 \quad CAB_1 \quad \dots \quad CA^{n-1}B_1 ) = q$$

*even if the system  $(S_1)$  is not controllable on  $[0, T]$ .*

*ii)  $B_1$  may dominates another operator  $B_2$ , with respect to  $C$  on  $[0, T]$ , without having*

$$\text{rank} ( CB_1 \quad CAB_1 \quad \dots \quad CA^{n-1}B_1 ) = q$$

*This is illustrated in the following example.*

**Example 2.7**

*i) We consider the case where  $n = 2$ ,  $p = q = 1$  and*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad C = ( 1 \quad 0 )$$

*The controllability matrix is given by*

$$( B_1 \quad AB_1 ) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

*its rank is then  $1 < 2$ . Consequently, the corresponding system is not controllable on  $[0, T]$ . On the other hand*

$$( CB_1 \quad CAB_1 ) = ( 1 \quad 1 )$$

*Its rank is  $1 = q$ , then  $B_1$  dominates any operator  $B_2$  on  $[0, T]$  with respect to  $C$ .*

*ii) Now, for  $m = n = 2$ ,  $p = 1$ ,  $q = 2$  and*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

*we have*

$$\begin{aligned}
 e^{A^*(T-s)}C^*\theta &= e^{(T-s)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \\
 &= e^{(T-s)} \begin{pmatrix} \theta_1 + \theta_2 \\ \theta_1 + \theta_2 \end{pmatrix}
 \end{aligned}$$

and  $B_1^*e^{A^*(T-s)}C^*\theta = e^{(T-s)}2(\theta_1 + \theta_2)$ , then

$$\| B_1^*e^{A^*(T-\cdot)}C^*\theta \|_{L^2(0,T;\mathbb{R})}^2 = 4 \int_0^T e^{2(T-s)}(\theta_1 + \theta_2)^2 ds$$

On the other hand

$$\begin{aligned}
 \| B_2^*e^{A^*(T-\cdot)}C^*\theta \|_{L^2(0,T;\mathbb{R}^2)}^2 &= \int_0^T e^{2(T-s)} \left\| \begin{pmatrix} \theta_1 + \theta_2 \\ \theta_1 + \theta_2 \end{pmatrix} \right\|^2 ds \\
 &= 2 \int_0^T e^{2(T-s)}(\theta_1 + \theta_2)^2 ds
 \end{aligned}$$

hence

$$\| B_2^*e^{A^*(T-\cdot)}C^*\theta \|_{L^2(0,T;\mathbb{R}^2)} \leq \| B_1^*e^{A^*(T-\cdot)}C^*\theta \|_{L^2(0,T;\mathbb{R})}$$

Consequently,  $B_1$   $C$ -dominates  $B_2$  on  $[0, T]$ , even if

$$\text{rank} \begin{pmatrix} CB_1 & CAB_1 \end{pmatrix} = \text{rank} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 1 \neq 2$$

□

In the following result, we give a necessary and sufficient condition for such a domination.

**Proposition 2.8**

$B_1$   $C$ -dominates  $B_2$  on  $[0, T]$ , if and only if

$$\text{Im} \begin{pmatrix} CB_2 & CAB_2 & \dots & CA^{n-1}B_2 \end{pmatrix} \subset \text{Im} \begin{pmatrix} CB_1 & CAB_1 & \dots & CA^{n-1}B_1 \end{pmatrix}$$

Proof : Using proposition 2.2,  $B_1$   $C$ -dominates  $B_2$  on  $[0, T]$ , if and only if

$$\text{Ker}(H_1^*C^*) \subset \text{Ker}(H_2^*C^*)$$

Using Caylay-Hamilton theorem, we deduce that for  $i = 1, 2$



$$y \in \text{Ker}[(H_i)^*C^*] \iff \begin{pmatrix} B_i^*C^* \\ B_i^*A^*C^* \\ \vdots \\ B_i^*(A^*)^{n-1}C^* \end{pmatrix}_{(np,q)} y = 0$$

hence

$$\text{Ker} \begin{pmatrix} B_i^*C^* \\ B_i^*A^*C^* \\ \vdots \\ B_i^*(A^*)^{n-1}C^* \end{pmatrix} = \text{Ker}(H_i^*C^*)$$

Consequently,  $B_1$   $C$ -dominates  $B_2$  on  $[0, T]$ , if and only if

$$\text{Im} \begin{pmatrix} CB_2 & CAB_2 & \dots & CA^{n-1}B_2 \end{pmatrix} \subset \text{Im} \begin{pmatrix} CB_1 & CAB_1 & \dots & CA^{n-1}B_1 \end{pmatrix}$$

□

### Remark 2.9

1. This necessary and sufficient condition is not depending on the time parameter  $T$ .
2. In the particular case where  $B_2$  is invertible,  $B_1$   $C$ -dominates  $B_2$  on  $[0, T]$  if and only if

$$\text{rank} \begin{pmatrix} CB_1 & CAB_1 & \dots & CA^{n-1}B_1 \end{pmatrix} = \text{rank} ( C )$$

## 2.2 Minimum energy problem

In this part, we assume that  $B_1$   $C$ -dominates  $B_2$  on  $[0, T]$ , then for any  $v \in L^2(0, T; \mathbb{R}^m)$ , there exists a control  $u \in L^2(0, T; \mathbb{R}^p)$  such that

$$CH_1u + CH_2v = 0 \tag{10}$$

For  $v \in L^2(0, T; \mathbb{R}^m)$ , we examine the existence and the uniqueness of the optimal control  $u \in L^2(0, T; \mathbb{R}^p)$  satisfying (10), i.e. ensuring the compensation of the opposing term  $B_2v$ .

For this, we use an extension of the Hilbert Uniqueness Method. Indeed, for  $\theta \in \mathbb{R}^q$ , let us note :

$$\|\theta\|_* = \left( \int_0^T \|(H_1)^* C^* \theta\|_{\mathbb{R}^p}^2 ds \right)^{\frac{1}{2}} = \left( \int_0^T \|B_1^* e^{A^*(T-s)} C^* \theta\|_{\mathbb{R}^p}^2 ds \right)^{\frac{1}{2}}$$

$\|\theta\|_*$  is a semi-norm on  $\mathbb{R}^q$ .

We assume that  $\|\cdot\|_*$  is a norm on  $\mathbb{R}^q$ . If  $\text{Ker} [(H_1)^* C^*] = \{0\}$ , this is equivalent to the asymptotic remediability [1,2,3] of the system (1)+(2) on  $[0, T]$ . The corresponding inner product is given by :

$$\langle \theta, \sigma \rangle_* = \int_0^T \langle B_1^* e^{A^*(T-s)} C^* \theta, B_1^* e^{A^*(T-s)} C^* \sigma \rangle ds$$

and the operator  $\Lambda_C : \mathbb{R}^q \rightarrow \mathbb{R}^q$  defined by

$$\Lambda_C \theta = CH_1(H_1)^* C^* \theta = \int_0^T C e^{(T-s)} B_1 B_1^* e^{A^*(T-s)} C^* \theta ds$$

is symmetric and positive definite, and then invertible. We give hereafter the expression of the optimal control ensuring the compensation of the effect of the term  $B_2 v$ , at the final time  $T$ .

**Proposition 2.10**

For  $v \in L^2(0, T; \mathbb{R}^m)$ , there exists a unique  $\theta_v \in \mathbb{R}^q$  such that

$$\Lambda_C \theta_v = -CH_2 v$$

and the control

$$u_{\theta_v}(\cdot) = B_1^* e^{A^*(T-\cdot)} C^* \theta_v$$

verifies

$$CH_1 u_{\theta_v} + CH_2 v = 0$$

Moreover, it is optimal and

$$\|u_{\theta_v}\|_{L^2(0, T; \mathbb{R}^p)} = \|\theta_v\|_*$$

Let us note that one can also consider a general optimal control problem with a cost function defined on  $L^2(0, T; \mathbb{R}^p)$  by

$$\begin{aligned}
J(u) &= \langle P(CH_1u + CH_2v), CH_1u + CH_2v \rangle \\
&+ \int_0^T \langle Q(CH_1(t)tu + CH_2(t)v), CH_1(t)u + CH_2(t)v \rangle dt \quad (11) \\
&+ \int_0^T \langle Ru(t), u(t) \rangle dt
\end{aligned}$$

where  $P$ ,  $Q$  and  $R$  are symmetric matrixes with  $Q$  positive,  $P$  and  $R$  are positive definite.

In the next section, we present an extension to the asymptotic case.

### 3 Extension to the asymptotic case

In this part, we consider a class of linear dynamical systems described by the following state equation

$$(S_1^\infty) \begin{cases} \dot{z}(t) = Az(t) + B_1u_1(t) + B_2u_2(t); & t > 0 \\ z(0) = z_0 \in \mathbb{R}^n \end{cases} \quad (12)$$

with  $A \in M_n(\mathbb{R})$ ,  $B_1 \in M_{n,p}(\mathbb{R})$ ,  $B_2 \in M_{m,n}(\mathbb{R})$ ,  $u_1 \in L^2(0, +\infty; \mathbb{R}^p)$  and  $u_2 \in L^2(0, +\infty; \mathbb{R}^m)$

The system (12) is augmented by the output equation :

$$y(t) = Cz(t); \quad t > 0 \quad (13)$$

with  $C \in M_{q,n}(\mathbb{R})$ . We have

$$z(t) = e^{At}z_0 + H_1(t)u_1 + H_2(t)u_2$$

Let

$$z = \begin{pmatrix} z_+ \\ z_- \end{pmatrix}$$

where  $z_+$  and  $z_-$  are respectively the projections of the state  $z$  on the unstable and the stable subspaces :

$$E_+ = \bigoplus_{R(e)(\lambda) \geq 0} Ker(A - \lambda I_n)^{m(\lambda)} \quad (14)$$

and

$$E_- = \bigoplus_{\text{Re}(\lambda) < 0} \text{Ker}(A - \lambda I_n)^{m(\lambda)} \tag{15}$$

where  $m(\lambda)$  is the multiplicity of the eigenvalue  $\lambda$ .  $E_+$  and  $E_-$  are invariant with respect to the operator  $A$ . We have

$$\begin{cases} (S^+) \quad \dot{z}^+(t) = A_+ z^+(t) + P B u_1(t) + P B u_2(t) \\ (S^-) \quad \dot{z}^-(t) = A_- z^-(t) + (I - P) B u_1(t) + (I - P) B u_2(t) \end{cases} \tag{16}$$

$P$  is the projection operator on the unstable part and  $A_+$ , respectively  $A_-$ , is the matrix induced by  $A$  on  $E_+$ , respectively  $E_-$ .

In the case where we observe only the stable part, i.e. if

$$E_+ \subset \text{Ker}(C)$$

the following operators  $K^\infty(C)$  and  $L^\infty(C)$  respectively given by

$$\begin{aligned} K^\infty(C) : L^2(0, +\infty; \mathbb{R}^p) &\longrightarrow \mathbb{R}^q \\ u &\longrightarrow \int_0^\infty C e^{At} B_1 u(t) dt \end{aligned}$$

and

$$\begin{aligned} L^\infty(C) : L^2(0, +\infty; \mathbb{R}^m) &\longrightarrow \mathbb{R}^q \\ v &\longrightarrow \int_0^\infty C e^{At} B_2 v(t) dt \end{aligned}$$

are well defined. We have the same result if the considered system is exponentially stable, i.e. the matrix  $A$  is such that  $\text{Re}(\lambda_i) < 0$  for  $i = 1, n$ ; where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$ . But as it will be shown later, this is not necessary.

We assume that operators  $K^\infty(C)$  and  $L^\infty(C)$  are well defined. By the same, we say that  $B_1$   $C$ -dominates  $B_2$  asymptotically, if for every  $u_2 \in L^2(0, +\infty; \mathbb{R}^m)$ , there exists  $u_1 \in L^2(0, +\infty; \mathbb{R}^p)$  such that :

$$K^\infty(C)u_1 + L^\infty(C)u_2 = 0$$

With a similar approach, it is easy to show the following characterization result of the asymptotic domination.

**Proposition 3.1** *The following properties are equivalent :*

i)  $B_1$   $C$ -dominates  $B_2$  asymptotically.

ii)  $Im [L^\infty(C)] \subset Im [K^\infty(C)]$

iii)  $Ker [K^\infty(C)] \subset Ker [L^\infty(C)]$

iv)  $\exists \gamma > 0$  such that  $\forall \theta \in \mathbb{R}^q$ , we have

$$\| B_2^* e^{A^* \cdot} C^* \theta \|_{L^2(0,+\infty;\mathbb{R}^m)} \leq \gamma \| B_1^* e^{A^* \cdot} C^* \theta \|_{L^2(0,+\infty;\mathbb{R}^p)}$$

v)

$$Im ( CB_2 \quad CAB_2 \quad \dots \quad CA^{n-1}B_2 ) \subset Im ( CB_1 \quad CAB_1 \quad \dots \quad CA^{n-1}B_1 )$$

Let us note that if the eigenvalues  $\lambda_1, \dots, \lambda_n$  of  $A$  are such that :  $Re(\lambda_i) < 0$  for  $i = 1, n$ , the proposition 2.4 remain true in the asymptotic case.

One can also consider the asymptotic optimal control problem. The approach and the results are similar.

We give hereafter illustrative examples showing particularly that the notions of stability (or even the stabilizability) and also the controllability are not necessary for considering the asymptotic domination.

### Example 3.2

*Let us consider the case of an unstable system with  $m = n = 2$ ,  $p = q = 1$  and*

$$A = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad C = ( 1 \quad -3 )$$

*Obviously, the system is not stable. We have*

$$e^{tA} = P e^{t\Delta} P^{-1}$$

*where*

$$\Delta = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}; \quad P = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$$

*then*

$$C e^{tA} = ( e^{-t} \quad -3e^{-t} )$$

In this case  $K^\infty(C)$  and  $L^\infty(C)$  are well defined and

$$(CB_1 \quad CAB_1) = (1 \quad -1)$$

and  $\text{rank}(C) = 1$ . Consequently,  $B_1$   $C$ -dominates  $B_2$  asymptotically.

□

**Example 3.3** We consider an unstable system with  $m = n = 2$ ,  $p = q = 1$  and  $B_2$  is invertible.

i) For

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad C = (1 \quad 1); \quad F = (a \quad b)$$

we have

$$A + B_1F = \begin{pmatrix} 1+a & b \\ 0 & 2 \end{pmatrix}$$

$(A, B_1)$  is not stabilizable because for any  $F = (a \quad b)$ ,  $A + B_1F$  is not stable. However,  $B_1$   $C$ -dominates  $B_2$  asymptotically.

ii) Let

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad C = (1 \quad 1)$$

For  $F = (a \quad b)$ , the matrix  $A + B_1F = \begin{pmatrix} -1 & 0 \\ a+1 & b+2 \end{pmatrix}$  is stable for  $b < -2$ , then  $(A, B_1)$  is stabilizable.

On the other hand, we have  $AB_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ , then  $\text{rank}(B_1 \quad AB_1) = 1 \neq 2$  and consequently  $(A, B_1)$  is not controllable (in fact, the asymptotic controllability is not well defined in the considered case). But  $B_1$   $C$ -dominates  $B_2$  asymptotically, because

$$\text{rank}(CB_1 \quad CAB_1) = \text{rank}(C) \tag{17}$$

Let us remark that in the general case, if  $C$  is an invertible matrix (for example if  $C$  is the identity matrix), then the relation (9) (the relation (17) in the considered example) is equivalent to the controllability rank condition. But this hypothesis is strong and is not very useful.

□

Finally let us note that concerning the asymptotic minimum energy problem, and with an extension of H.U.M., it is easy to show the existence and the uniqueness of the optimal control and also how to find it.

With convenient hypothesis, one can also consider the asymptotic version of the cost function given by (11).

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## Références

- [1] L. Afifi, A. Chafiai and A. El Jai (2001)  
*Spatial Compensation of boundary disturbances by boundary actuators* the International Journal of Applied Mathematics and Computer Science, Vol. 11, N°4.
- [2] L. Afifi, A. El Jai and E. M. Magri (2008)  
*Compensation problem in finite dimension linear dynamical systems*. International Journal of Applied Mathematical Sciences. Vol.2, N.45, pp 2219 - 2228.
- [3] L. Afifi, M. Bahadi, A. El Jai et A. El Mizane (2009) *Asymptotic Analysis, Approximations and Simulations of the Compensation Problem in Hyperbolic systems*. Applied Mathematical Sciences, volume 3, N°15, 737-765.
- [4] L. Afifi, E. M. Magri and A. El Jai (2010)  
*Weak and exact domination in distributed systems*. To appear in the International Journal of Applied Mathematics and Computer Science. Vol. 20, No. 3.
- [5] R.F. Curtain and A.J. Pritchard (1978)  
*Infinite Dimensional Linear Systems Theory*. Lecture Notes in Control and Information Sciences, vol. 8, Berlin.
- [6] Curtain, R.F. and Zwart, H.J. (1995). *An Introduction to Infinite-Dimensional Linear Systems Theory*. Texts in Applied Mathematics, Springer-Verlag, New York.
- [7] J.-L. Lions (1980)  
*On the foundation of optimal control of distributed systems*. Lecture notes in control and inf. sc. 22, Springer.
- [8] J.-L. Lions (1988)  
*Contrôlabilité exacte, perturbations et stabilisation des systèmes distribués*. Paris, Masson.

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